THE UNIVERSITY OF ALBERTA

THE RELATIVISTIC BOLTZMANN EQUATION

by

© JOHN NICHOLAS VARDALAS

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled THE RELATIVISTIC BOLTZMANN EQUATION submitted by JOHN NICHOLAS VARDALAS in partial fulfillment of the requirements for the degree of Master of Science.

ABSTRACT

In the early chapters (1-3) an expository review is given of Classical and Relativistic Kinetic Theory of simple gases, where the Relativistic Kinetic Theory, as mainly developed by J.L. Synge ["The Relativistic Gas", North Holland Publishers] and W. Israel ["Relativistic Kinetic Theory of a Simple Gas", Journal of Math. Phys. Vol. 4, No. 9], is that of a simple self-gravitating gas of neutral particles with no other external forces undergoing elastic binary collisions and where the "mean-free path" is space-time. In the early part of Chapter 4 small deviations from equilibrium are discussed for the simple gas along with certain phenomenological considerations. In tha latter part of Chapter 4 new explicit expressions for the transport coefficients (i.e. Conductivity, Bulk viscosity, and Shear viscosity) of the simple relativistic gas are given. In Chapter 5 certain consequences of the existence of Bulk viscosity and its asymptotic temperature dependence are briefly discussed.

ACKNOWLEDGMENT

I would like to express my appreciation to Professor W. Israel for suggesting the problem in the latter part of this thesis and for the many enjoyable and illuminating discussions concerning the topics in this thesis, and, Relativity in general.

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CHAPTER I

Equilibrium State

In this chapter a brief discussion of the statistical nature of the equilibrium configuration of a simple gas is discussed both for the Classical and Relativistic cases.

Given a gas consisting of n particles where n is quite large, we would like to describe its macroscopic behavior in terms of its microscopic variables. To present a precise account of its behavior would clearly be a gargantuan task because of the enormity of the initial data and variables involved. This leads one to a statistical approach to the problem. In this section the usual statistical approach used in the derivation of the Maxwell-Boltzmann distribution function will be discussed for the Classical and Relativistic cases.

The statistical approach takes the form of specifying some distribution function. Instead of knowing each particle's exact behavior i.e. its position and momentum, we will now know, via the distribution function, that a given number of particles have, for example, a momentum within a certain range centered about some value of momentum. In the classical case the space in which we operate in a 6-dimensional one. It is called the μ-space. Even though each particle has 6 degrees of freedom (excluding spin effects, etc.), and there are n of these particles, we superimpose them all one one 6-dimensional grid. Mathematically

then we have some distribution function $N(\underset{\sim}{r},\underset{\sim}{v},t)$ which is a function of position, velocity, and time. Given an elemental region in our μ-space, i.e. d^3rd^3v (1) about some point $(\underset{\sim}{r},\underset{\sim}{v})$, then $N(\underset{\sim}{r},\underset{\sim}{v},t)d^3\ rd^3v$ represents the number of particles that have coordinate values that fall within the elemental volume d^3rd^3v at time t . It is in this setting that the Boltzmann Transport equation is obtained. But in order to get a more fundamental insight into what the equilibrium configuration means statistically, we will enlarge our space from a 6-dimensional one to a 6n-dimensional space. This new space is called the Γ-space. In the μ-space representation n points determine the state of the gas. But in Γ space one point specifies the whole system. Suppose we interchange two particles from two cells. This would still give us the same configuration in μ-space, i.e., the same N , but when this interchange is viewed in Γ-space it leads to a new distinct point. Because of this we see that, associated with a given distribution function N , or configuration in μ space, we have a multiplicity, or a cloud, of points in Γ space. In fact this multiplicity should be expected, because we should really be starting in the Γ-space, and since μ-space is a kind of projection of Γ space down onto one 6 -dimensional grid, one expects to lose information in the process.

The equilibrium state for a gas will now be associated with that state that maximizes the "volume" that the system takes up in Γ space. What we are saying is that, given a system with certain macroscopic limitations (e.g. fixed energy, number, etc.), the equilibrium state will be that which allows for the greatest number of permutations of the microscopic

variables and yet leaves the macroscopic system unchanged (2). This is equivalent to saying that our system will have the most probable distribution.

The problem therefore becomes one of maximizing a volume or maximizing the probability associated with any given distribution. The approach that will be taken is that of maximizing the probability P, where P stands for the probability of having a given distribution. Thus we essentially can stay in μ-space.

Divide up μ-space into elemental cells all having the same magnitude of extension, and index them by $1,2,\cdots,i,\cdots$. Let us now take the n particles and distribute them out amongst the cells, where we write n_i as the number of particles in the ith cell. But the particles can't distribute themselves in any fashion, they must first satisfy,

$$\sum_i n_i = n \text{ .} \tag{1-1}$$

Since we are assuming that the total energy is fixed then we must also satisfy,

$$\sum_i n_i \epsilon_i = E \quad , \tag{1-2}$$

where ϵ_i is the energy per particle in the ith cell. It is a well known fact from combinatorics that the probability for a given distribution (3), for fixed population numbers $n_1,\cdots,n_i,\cdots$, is

(1-3) $$P = \frac{n!}{n_1!n_2!\cdots n_i!} \times \text{const.} \quad .$$

Taking the log of equation 1-3, we get,

(1-4) $$\log P = \log (n!) - \sum \log (n_i!) + \text{const.} \quad .$$

Assuming that n_i are all quite large we can make use of the approximation

(1-5) $$\log (n!) = n \log n - n \quad ,$$

which follows from Sterling's formula. Substituting equation 1-5 into equation 1-4, and noting equation 1-1 we get,

(1-6) $$\log P = n \log n - \sum_i n_i \log n_i + \text{const.} \quad .$$

Since the most probable distribution means it is more probable relative to any other distribution, let us then compare any two distributions of n particles to obtain their relative probability. By means of equation 1-6 we see that,

(1-7) $$\log(P/P') = - \sum_j n_j \log n_j + \sum_j n'_j \log n'_j$$

where n_j and n'_j are the occupation numbers of the two distributions. Letting

(1-8) $$Q = - \sum_j n_j \log n_j \quad , \quad Q' = - \sum_j n'_j \log n'_j$$

it can be said that the distribution $\{n_1, \cdots, n_j \cdots\}$ is more probable than the distribution $\{n_1', \cdots n_j', \cdots\}$ i.e. $P > P'$ if $Q > Q'$. Similarly, $P' > P$ if $Q' > Q$, and $P = P'$ if $Q' = Q$. In comparing two distributions there are two conditions which are required. Firstly, the total number of particles must be the same. Secondly, the division of μ-space into elemental cells must be done in the same fashion for both distributions. Since we have divided μ-space into cells of equal extension we label the magnitude of extension as $\epsilon\eta$, where ϵ is the spatial volume, η the velocity (or equivalently the momentum) volume. We can thus write

(1-9) $$n_j = \epsilon\eta\, N_j \quad ,$$

where N_j is the value of the distribution function over the jth cell. Using equation 1-9 the expression for Q can be rewritten as

(1-10) $$Q = -\,\epsilon\eta \sum N_j \log N_j - \epsilon\eta\; n \log \epsilon\eta \quad .$$

Going back to equations 1-7, 1-8, and in view of equation 1-10, it can be seen that when probabilities are compared, we are essentially comparing expressions of the form

(1-11) $$F \equiv -\sum N_j\,(\log N_j)\,\epsilon\eta \; .$$

F is therefore a measure of the relative probability. Demanding the usual conditions, equation 1-11 can be rewritten as

(1-12) $$F \equiv - \int N \log N \, d^3r \, d^3v \quad ,$$

where ϵ , and η have become d^3r , and d^3v respectively. This quantity F , which is a measure of the chaos of the system, is called the entropy integral. Considering some fixed elemental volume d^3r , over which F will be maximized, the equilibrium condition is

(1-13) $$\delta F = 0 \quad ,$$

where the previous constraints (equations 1-1,1-2) for the continuous case are

(1-14) $$n = \int N \, d^3r \, d^3v \qquad \text{where} \qquad \delta n = 0$$
$$E = \int N \epsilon \, d^3r \, d^3v \qquad \text{where} \qquad \delta E = 0$$

Varying N with δN arbitrary, and using the above constraints, equation 1-14 becomes

(1-15) $$(\log N + 1)\delta N = A\delta N + \beta\epsilon\delta N$$

where A and β are constants. Since the δN are arbitrary equation 1-15 can be put into the form

(1-16) $$N = C \, e^{-\beta\epsilon} \quad .$$

It can be shown that $C = n(\frac{m}{2kT})^{3/2}$ and $\beta = \frac{1}{kT}$ (4). Recalling that $\epsilon = \frac{p^2}{2m}$, we see that equation 1-16 is the usual Maxwell-Boltzmann distribution.

In the Relativistic case the equilibrium condition is arrived at by essentially the same methods as were used for the Classical case. That is, the equilibrium condition is characterized by $\delta F = 0$, where F has the same form as the classical case. But in the Relativistic case, the formulation of μ-space needs closer scrutiny. Also this approach must make manifest its covariance. Before the relativistic derivation can be given there are certain preliminaries which must first be given. We say that space-time forms a 4-dimensional manifold. Each point in this manifold is called an event, labelled x^μ , where the global space-time structure need not be Minkowskian. Consider a particle, and at an event x^μ associated with the particle is the 4-momentum P^μ . The 4-momentum of the particle lies in the "plane" tangent at x^μ to space-time. We call this plane the momentum space. The momentum space is Minkowskian. At the event x^μ we can form a tetrad in Momentum space. The vectors in the tetrad are tangent to the coordinates net in space-time at event x^μ . Since all of our work will be carried on in a local Minkowskian frame in space-time we see then that each vector in the momentum tetrad is parallel to a corresponding vector in the tetrad we pick in space-time.

Of all those vectors that exist in momentum space, the only ones of interest will be the future pointing, time-like vectors, i.e. $P_\mu P^\mu = -m^2$, and $P^4 > 0$. Since in our case we will be concerned with

the situation where all the particles have the same constant proper mass it is seen that $P_\mu P^\mu = -m^2$ describes a pseudosphere (5) of radius m . Thus all the particles in the system have a 4-momentum vector which terminates on the pseudosphere of radius m . Polar coordinates can be introduced:

$$
\begin{aligned}
P^1 &= m \sinh \chi \;\sin \theta \;\sin \phi \\
P^2 &= m \sinh \chi \;\sin \theta \;\sin \phi \\
P^3 &= m \sinh \chi \;\cos \theta \\
P^4 &= m \cosh \chi
\end{aligned}
\qquad , \tag{1-17}
$$

where $0 \leq \chi < \infty$, $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$, (if we were admitting variable proper masses then $0 < m < \infty$) .

In the classical case the distribution function N is a function of $\underset{\sim}{r}, \underset{\sim}{v}$, and t , and N is defined such that

$$\nu = N(\underset{\sim}{r},\underset{\sim}{v},t)\; d^3r\; d^3v$$

represents the number of particles an observer would see if he constructed a volume d^3r about a point $\underset{\sim}{r}$ and counted the number of particles whose velocity vectors were within a range d^3v of $\underset{\sim}{v}$. Because of the fact that in Newtonian Theory "space" degenerates into a plane, our volumes have an absolute character. Butthis doesn't quite work out in Relativity. Depending on how the time-like unit vector is chosen, there will be many 3 flats orthogonal to the normal. It should be pointed out that momentum space will be used instead of velocity space (6).

The volume that replaces the classical volume d^3v , will be a 3-cell lying on the pseudosphere, so that given the 4-momentum P^μ of a particle it will terminate on the pseudosphere and all the particles with 4-momemta that lie within a given range of P^μ will terminate on an element of surface of the pseudosphere. Pictorically we have Figure 1, where A is a 3-cell about a point on the pseudosphere.

To see how the distribution function is defined within the Relativistic framework, let us consider at some event x^μ a time-like unit vector n_μ and orthogonal to n_μ an elemental 3-flat ds which will serve as our target. We want to know how many particles there are with the properties that their 4-momenta lie within a given elemental range centered about some given momentum P^μ and that they cross ds . To take into account the fact that n^μ is not coincident with P^μ , and hence our 3-flat ds is not normal to P^μ , we project the elemental 3-area on the pseudosphere along n_μ down onto a 3-flat parallel to ds . Let this projected 3-volume be called $d\Omega$. Pictorically we have figure 2.

Noting that $(\frac{P_\mu}{m})(\frac{P^\mu}{m}) = -1$ describes the surface of the pseudosphere, then by differentiating we get

$$(\frac{P^\mu}{m})\ d(\frac{P_\mu}{m}) = 0 \quad .$$

Therefore $(\frac{P^\mu}{m})$ is orthogonal to the tangent on the pseudosphere, hence $(\frac{P^\mu}{m})$ is orthogonal to the pseudosphere. Denoting the cell A by $md\omega$, then $d\Omega$ and $d\omega$ can be related by means of the Projection theorem

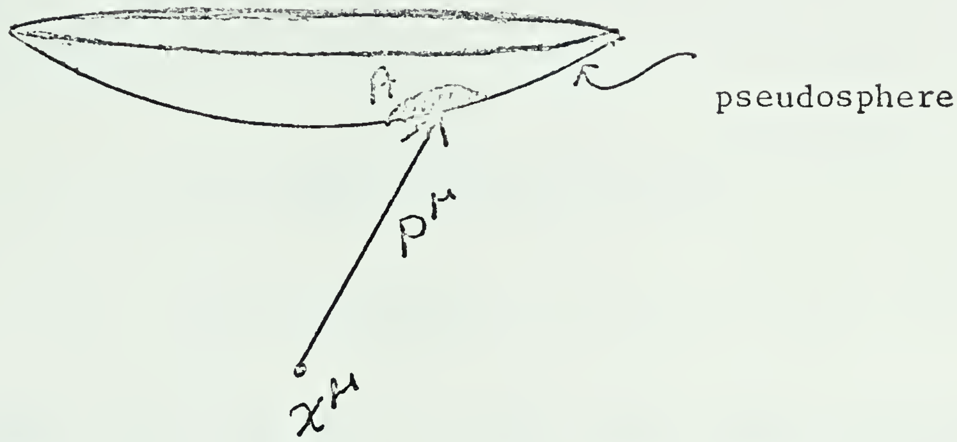


FIGURE 1

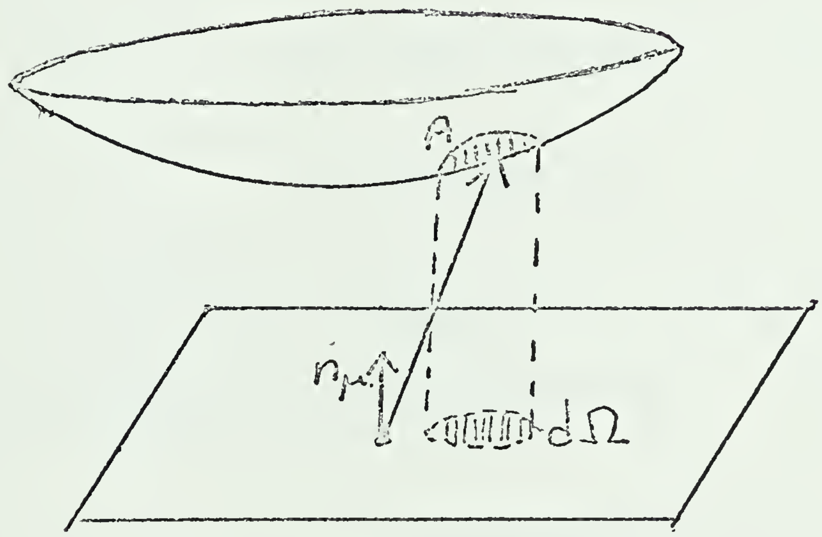


FIGURE 2

(7). That is,

$$d\Omega = m d\omega \left|\frac{P^\mu}{m} n_\mu\right| \quad , \quad \text{or} \tag{1-18}$$

$$d\Omega = d\omega \left|P^\mu n_\mu\right| \quad .$$

To relate $d\Omega$ and $d\omega$ in terms of our polar coordinates (i.e. equation 1-17), we use n_μ as our axis off which χ is measured (i.e. $\chi = 0$ along n_μ).

Since n_μ lies along the axis, thus $n_1 = n_2 = n_3 = 0$ and $n_4 = -1$. Equation 1-18 therefore becomes

$$d\Omega = d\omega \left|- m \cosh \chi\right| \quad \text{or ,} \tag{1-19}$$

$$d\Omega = m \cosh \chi \; d\omega \quad .$$

In the 3-flat orthogonal to n_μ we are just using normal spherical coordinates, where the radius is

$$R = m \sinh \chi \quad . \tag{1-20}$$

An element of volume in spherical coordinates is

$$d\Omega = R^2 \sin \theta \; d\, R \; d\theta \; d\phi \quad . \tag{1-21}$$

Substituting equation 1-20 into equation 1-21 we get

$$d\Omega = m^3 \sinh^2 \chi \cosh \chi \sin \theta \; d\chi \; d\theta \; d\phi \quad . \tag{1-22}$$

Therefore by equation 1-19 we get,

(1-23) $$d\omega = m^2 \sinh^2\chi \sin\theta \, d\chi \, d\theta \, d\phi \quad .$$

The quantity $d\omega$ is called the absolute 2-content of a 3-cell on the pseudosphere. The terminology "2-content" follows from the fact that $d\omega$ has dimensions of $(\text{Mass})^2$, which should not seem strange since we identified the volume of a 3-cell as $md\omega$ which we know has dimensions $(\text{Mass})^3$.

For the sake of future reference the relationship of $d\omega$ to the elementary solid angle in momentum space will be discussed. This will be of use when discussing the problem of scattering, where we want to know how many particles are scattered into a given solid angle (i.e. a given momentum range). Normally a solid angle is defined to be equal in magnitude to the surface area it subtends on a unit sphere. Similarly, given a unit pseudosphere, i.e. $m = 1$, then by equation 1-23 we get

(1-24) $$d\omega = \sinh^2\chi \sin\theta \, d\chi \, d\theta \, d\phi \quad .$$

So $\sinh^2\chi \sin\theta \, d\chi \, d\theta \, d\phi$ becomes our elementary solid angle. Also $d\omega$ now represents the numerical value of the volume of a 3-cell on the pseudosphere since $md\omega = d\omega$. If we don't have a unit mass then the elementary solid angle is $\frac{d\omega}{m^2}$. Many authors (8) make use of the simplicity of equation 1-24 by dealing with the normalized momentum vectors, i.e. $(\frac{P^\mu}{m})$, instead of P^μ. This has the effect of operating on the unit

pseudosphere.

Getting back to the distribution function N, we define it such that

$$\nu = N(\underset{\sim}{x},P)\ ds\ d\Omega \quad ,$$

represents the number of world lines that intersect ds with normal n_μ and such that the 4-momenta fall within the 3-cell $d\Omega$, where $d\Omega = |n_\mu P^\mu| d\omega$. $d\omega$ is an invariant but $d\Omega$ dpends on the choice of n_μ. It turns out that $N(x,P)$ is an invariant i.e. it does not depend on which 3-flat we take our sample populations from. The proof of the invariance of N rests essentially on two points. The first, which has already been mentioned, is the invariance of $d\omega$. The second point is that if we look at the same configuration of particles with respect to two different normals n_μ and n'_μ, then $ds\ d\Omega = ds' d\Omega'$. What we are saying is that the product $ds\ d\Omega$ is also an invariant. The reason for stipulating a fixed configuration is that if our comparison of two distribution functions is to be meaningful, then we should be referring to the same set of particles. This demand will be guaranteed by the way we construct our elemental 3-flats $ds\ d\Omega$, ds' and $d\Omega'$.

Consider a local tube of world lines characterized by momentum P^μ at event x^μ, (see figure 3). Let n_μ be some time-like unit vector at event x^μ, and let ds be a cross section of the world-tube normal to n_μ, (see figure 4), and suppose ν is such that

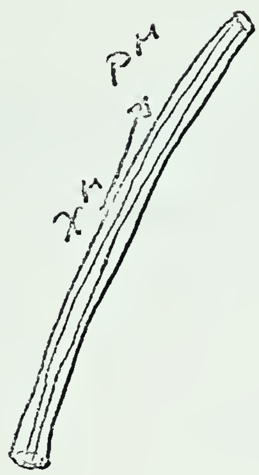


FIGURE 3

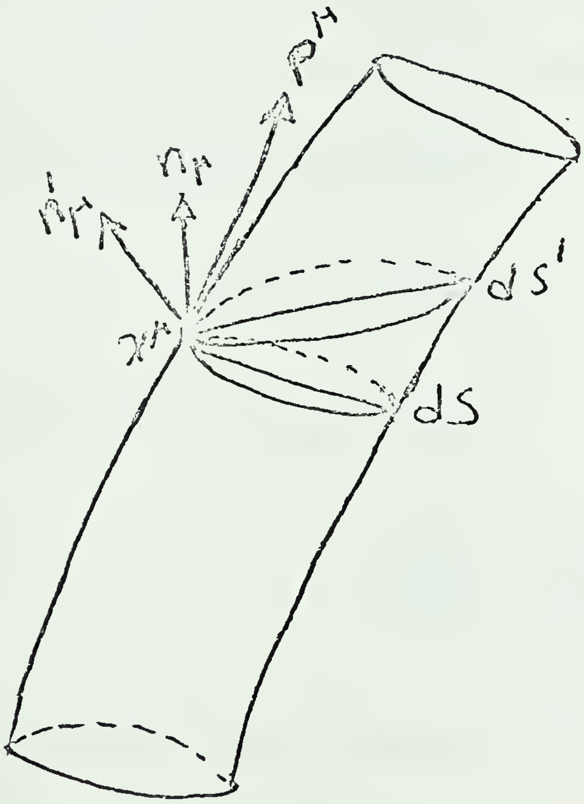


FIGURE 4

(1-25) $$\nu = N(x,P)\ ds\ d\Omega \quad .$$

Consider another time-like unit vector n'_μ at event x^μ, and let ds' be a cross section of this same tube normal to n'_μ then suppose we have

(1-26) $$\nu' = N'(x,P)\ ds'\ d\Omega' \quad .$$

We can now guarantee that we have the same configuration of particles if we say that $d\Omega'$ and $d\Omega$ correspond to the same 2-content $d\omega$. This is we can start with $d\Omega$ and obtain our $d\Omega'$ by projecting $d\Omega$ up to $d\omega$ along n_μ and then projecting down along n'_μ to obtain $d\Omega'$. Pictorically, we have figure 5. By the projection theorem we have

(1-27) $$ds'\ |P^\mu n'_\mu| = ds\ |P^\mu n_\mu| \ ,$$

and we also have by equation 1-18

(1-28) $$d\omega = \frac{d\Omega'}{|P^\mu n'_\mu|} = \frac{d\Omega}{|P^\mu n_\mu|} \quad .$$

Equations 1-27, 1-28 yield the result

(1-29) $$ds\ d\Omega = ds'\ d\Omega' \quad .$$

Since we are dealing with the same particle, $\nu = \nu'$, so that using equation 1-29, equations 1-25, 1-26 yield $N = N'$. This tells us that the distribution function N is independent of our choice

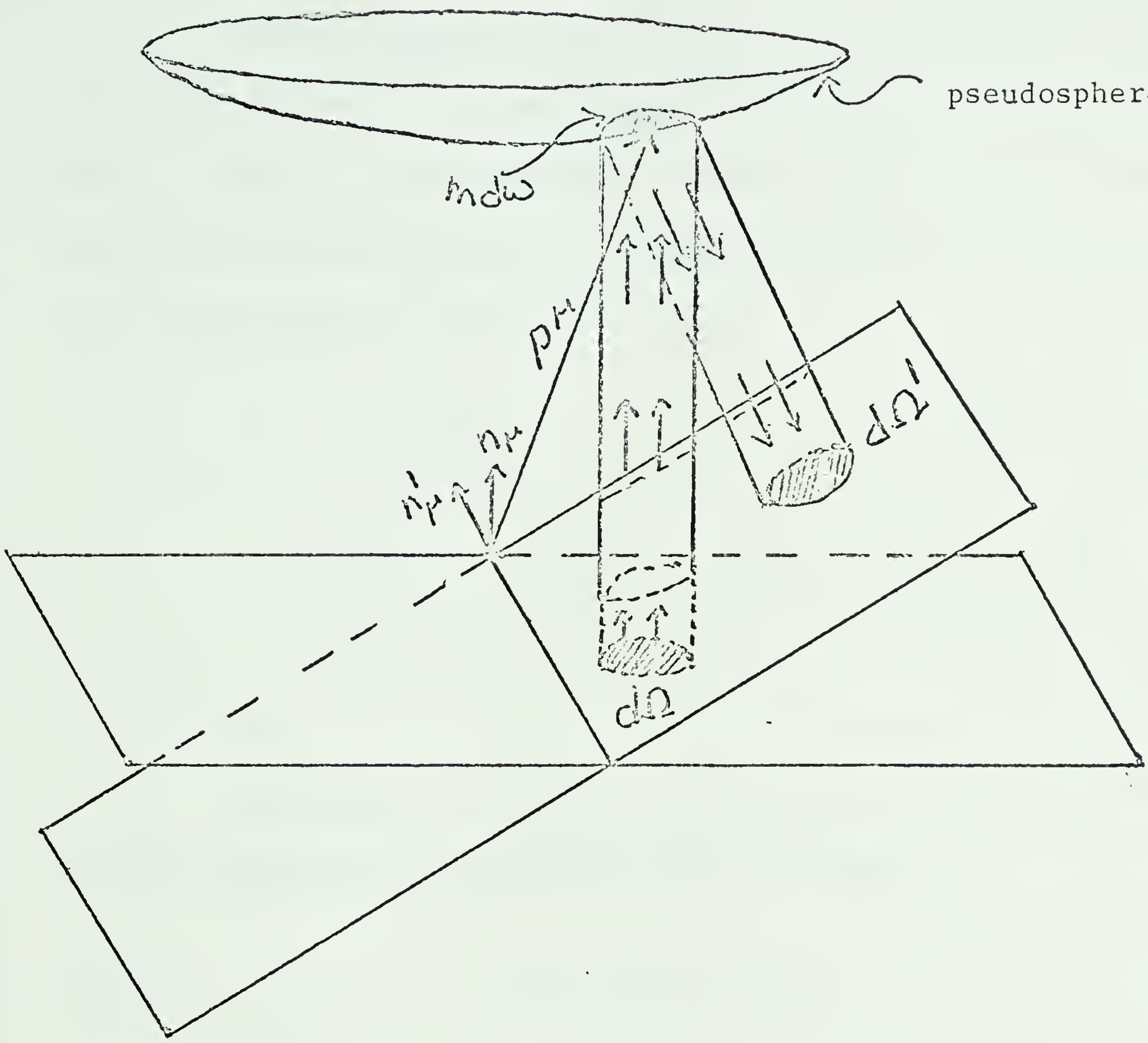


FIGURE 5

of n_μ .

As was mentioned earlier the derivation of the equilibrium distribution function is analogous to the classical case. But the cells ϵ and η now represent elemental cells in the 3-flats of space-time and momentum space respectively, where they are normal to some given timelike unit vector, i.e.

By arguments similar to those used for the classical case, we will maximize the following entropy integral

(1-30) $$F \equiv \int N \log N \, ds \, d\Omega \quad ,$$

and the condition for equilibrium will be

(1-31) $$\delta F = 0 \quad .$$

The only thing left is to state the constraints. In the classical case we could vary N as long as in so doing we didn't violate the conditions of constant number, and energy. As it turns out the relativistic constraints are of a local nature. That is, the fixed quantities

(or predeterminations) that must be conserved when varying N are specified locally. To establish what these predeterminations might be we make use of a rather ingenious construction due to J.L. Synge (9). The key to this construction is that it utilizes the time-like character of the 4-momenta of the particles. We pick our target by considering at some event x^{μ} a future pointing null cone, and we consider a cross section, s, of the null cone which is normal to some time-like unit vector n_{μ}. Let $\sum$ be the region of the null cone between the event x^{μ} and the cross section s.

The beauty of this construction is that the world line of a particle crossing $\sum$ must then cross s, and, as long as we have conservation of numbers in a collision, then collisions in the space-time volume bounded by $\sum$, and s will not alter this fact, i.e. world lines cannot leak out of this volume (see figure 6). Because of the conservation of 4-momenta at each collision we see that, given a momentum flux across $\sum$, then s must have the same momentum flux.

Our constraints therefore take the form guaranteeing the relationship between the world lines, and the momenta crossing $\sum$, and s. Everything that has been said up to this point is not necessarily of a local nature. But now we want to consider an infinitesimal target ds so that s must be close to x^{μ}. In fact we assume that the distribution function N, the numerical flux vector $M^{\mu} = \int P^{\mu} N \, d\omega$, and the energy-momentum tensor $T^{\mu\nu} = \int P^{\mu} P^{\nu} N \, d\omega$ are smooth continuous functions so that their values on ds are approximately those at x^{μ}. It is at this stage

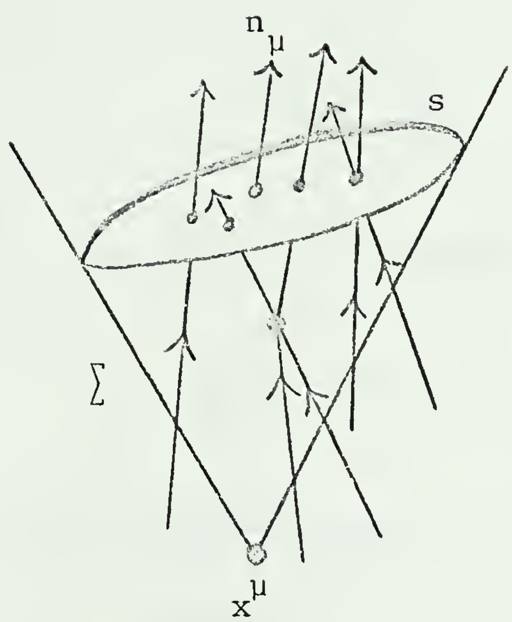


FIGURE 6

that the argument becomes one of a local nature, since N, M^μ, and $T^{\mu\nu}$ are evaluated at x^μ.

Suppose that the number of world lines crossing Σ is some fixed quantity ν. Then this must also be the same number crossing ds. Since the number of world lines crossing ds is $-M^\mu n_\mu \, ds$ then we can write

$$\nu = - M^\mu n_\mu \, ds . \tag{1-32}$$

Since $M^\mu = \int P^\mu N \, d\omega$, equation 1-32 becomes

$$\nu = ds \int - P^\mu n_\mu N \, d\omega . \tag{1-33}$$

But P^μ and n^μ are both time-like which means that $-P^\mu n_\mu > 0$, so that equation 1-33 can be rewritten as

$$\nu = ds \int N \, |P^\mu n_\mu| \, d\omega .$$

By equation 1-18 the above becomes

$$\nu = ds \int N \, d\Omega . \tag{1-34}$$

Now suppose that the momentum flux crossing Σ is some fixed quantity, say M^μ. But the momentum flux across ds is $(-T^{\mu\nu} n_\nu \, ds)$. Since $T^{\mu\nu} = \int P^\mu P^\nu N \, d\omega$, we can write

(1-35) $$- T^{\mu\nu} n_\nu \, ds = ds \int P^\mu \, |P^\nu n_\nu| \, d\omega = ds \int P^\mu \, N \, d\Omega \quad .$$

Therefore we can write,

(1-36) $$M^\mu = ds \int N \, P^\mu \, d\Omega \quad .$$

Thus the variations of N must be such that they are constraints (or predeterminations), equations 1-34, 1-35 are not violated.

Using lagrangian multipliers, the condition $\delta F = 0$ reads

(1-37) $$(\log N + 1)\delta N = c \, \delta N + \beta_\mu \, P^\mu \delta N$$

where c and β_μ are independent of P^μ, but are in general functions of x^μ. Equation 1-39 becomes

$$(\log N - c - \beta_\mu \, P^\mu + 1)\delta N = 0$$

which implies that

$$N = e^{c-1} \, e^{\beta_\mu P^\mu} \quad .$$

Letting $e^{c-1} \equiv \alpha(x)$ be finally get

(1-38) $$N(x,p) = \alpha(x) \, e^{\beta_\mu P^\mu} \quad ,$$

where β_μ must be time-like to guarantee convergence. Using equation 1-38 it can be shown that we get the numerical flux, and energy momentum

tensor of a perfect fluid with the hydrodynomic 4-velocity properly defined. But we save this for a later section where the conservation laws will be derived in a natural way from the Boltzmann equation.

CHAPTER II

Boltzmann Collision Equation

In this chapter the usual derivation of the Boltzmann Collision Equation is given for both the Classical and Relativistic Cases.

In the previous chapter we neglected the more general case of non-equilibrium. If we have a non-equilibrium situation we know it will eventually reach equilibrium, (i.e. the distribution function will approach the Maxwell-Boltzmann distribution), but to this point we cannot describe what the behavior of the system will be. For example it is in this approach to equilibrium where the transport effects occur as a means for smoothing out the non-uniformities responsible for the non-equilibrium situation. It is at this stage that the Boltzmann transport equation is introduced to explain the more general behavior of the gas. The Boltzmann transport equation describes the evolution of the distribution function by taking into account the effect collisions have on N. To make this clearer, we will give the following rather crude argument which will serve both to illustrate the above statement, and to introduce the Boltzmann transport equation.

Suppose that at $\underset{\sim}{r},\underset{\sim}{v}$ we have some hypothetical observer at time t. About him we have the volume $d^3r\, d^3v$. Suppose he observes some density distribution N of particles in this volume. Further suppose we stipulate that there are no collisions. We can view the time evolution

of these points as a mapping of μ-space into μ-space. After an infinitesimal time dt our "observer" has moved with the volume from $\underset{\sim}{r}, \underset{\sim}{v}$ at time t to

$$\underset{\sim}{r} + \frac{d\underset{\sim}{r}}{dt}\, dt \quad , \quad \underset{\sim}{v} + \frac{d\underset{\sim}{v}}{dt}\, dt \quad ,$$

at a time $t + dt$, and the volume has become $d^3r'\, d^3v'$. But by Liouvilles theorem $d^3r\, d^3v = d^3r'\, d^3v'$. Since we are talking about the same particles, we see that, under the assumption of no collisions, our observer will see no change in the density of points. That is,

$$\frac{dN}{dt} = 0 \quad , \text{ where } \frac{d}{dt} \text{ is the total derivative.} \tag{2-1}$$

Since N is a function of $\underset{\sim}{r}, \underset{\sim}{v}$, and t, we write equation 2-1 as,

$$\frac{dN}{dt} = \frac{\partial N}{\partial t} + \frac{\partial N}{\partial \underset{\sim}{r}} \cdot \frac{d\underset{\sim}{r}}{dt} + \frac{\partial N}{\partial \underset{\sim}{v}} \cdot \frac{d\underset{\sim}{v}}{dt} = 0 \qquad \text{or,}$$

$$m\frac{\partial N}{\partial t} + P \cdot \frac{\partial N}{\partial \underset{\sim}{r}} + \underset{\sim}{F} \cdot \frac{\partial N}{\partial \underset{\sim}{v}} = 0 \quad . \tag{2-2}$$

What we are saying is that, under the condition of no collisions, the density per unit "volume" of particles or points in μ-space, behaves like an incompressible fluid. But if we allow for close binary collisions then there will be particles which will leak out our volume due to these collisions. If some leak out of one elemental volume they then must leak into another. Therefore, in general, our observer moving with the volume will not find equation 2-2 satisfied, but instead he will observe a change

in N due to a net change in the number of particles in the elemental volume. To take this into account we modify equation 2-2 by adding the customary phenomenological collision term, i.e. for collisions equation 2-2 becomes

$$(2\text{-}3) \qquad \frac{\partial N}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial N}{\partial \underset{\sim}{r}} + \frac{\underset{\sim}{F}}{m} \cdot \frac{\partial N}{\partial \underset{\sim}{v}} = J_{gain} - J_{loss} \quad ,$$

where $J_{gain} - J_{loss}$ gives us the net gain or loss of particles per unit volume in μ-space. Equation 2-3 is called the Boltzmann Transport equation. The phenomenological term $J_{gain} - J_{loss}$ is explicitly obtained by considering the dynamics of elastic binary collisions and thus obtaining a differential cross section.

Since all the particles have the same mass the conservation of momentum law reads,

$$(2\text{-}4) \qquad \underset{\sim}{v} + {}'\underset{\sim}{v} = \underset{\sim}{v}^* + {}'\underset{\sim}{v}^*$$

where $\underset{\sim}{v}$ and $'\underset{\sim}{v}$ are the initial velocities and $\underset{\sim}{v}^*, 'v^*$ are the final velocities in a binary collision of two particles.

We are also considering only elastic collisions therefore,

$$(2\text{-}5) \qquad |\underset{\sim}{v}|^2 + |'\underset{\sim}{v}|^2 = |\underset{\sim}{v}^*|^2 + |'\underset{\sim}{v}^*|^2 \quad .$$

In the lab frame the collision looks like

By translating our reference frame to a velocity $\underset{\sim}{V} = \underset{\sim}{v} + {}'\underset{\sim}{v}$, and letting

$$(2\text{-}6) \qquad \underset{\sim}{g} = \underset{\sim}{v} - {}'\underset{\sim}{v} \quad \text{and} \quad \underset{\sim}{g}^{*} = \underset{\sim}{v}^{*} - {}'v^{*}$$

we can view the collision in the center of mass frame, where $\underset{\sim}{g}$ is the relative velocity of the incoming particles, and $\underset{\sim}{g}^{*}$ is the relative velocity of the outgoing particles. Because it is an elastic collision we know that

$$(2\text{-}7) \qquad |\underset{\sim}{g}| = |\underset{\sim}{g}^{*}| \quad .$$

Pictorically, we have figure 2, where θ , and ϕ are the angles between $\underset{\sim}{g}$ and $\underset{\sim}{g}^{*}$.

Since the motion of the two particles before and after the collision is symmetrical about the center of mass 0 , we can replace the two particle scattering problem by an equivalent one particle scattering problem, with a fixed center located at the center of mass. We

use spherical coordinates Θ , and ϕ , with Θ measured off the axis through 0 , parallel to $\underset{\sim}{g}$, where the incoming particle has velocity $\underset{\sim}{g}$ and scatters off with velocity $\underset{\sim}{g}^*$ (see figure 3).

The calculation of the collision term $J_{gain} - J_{loss}$ will be done by considering an elemental volume d^3r about the point $\underset{\sim}{r}$, and focussing our attention on those molecules with a velocity that is within d^3v about velocity $\underset{\sim}{v}$. We will say that a particle with velocity in the range d^3v , about $\underset{\sim}{v}$, will leave it if it suffers a collision with another particle, i.e. we have assumed that d^3v is small enough that a particle in that range, suffering a collision, will leave it. In the spatial volume d^3r there are also molecules in the range $d^3{'v}$, about some velocity $'\underset{\sim}{v}$, which act as an incident beam of particles on those with velocities in the range d^3v , about $\underset{\sim}{v}$. The flux of the incident beam is

(2-8) $$N(r,'v,t)\ d^3{'v}\ |\underset{\sim}{g}| \quad .$$

The number of particles that serve as "scattering center" in d^3r is

(2-9) $$N(\underset{\sim}{r},\underset{\sim}{v},t)\ d^3v \quad .$$

Define the quantity $\sigma(v,'v:v^*,'v^*)$ such that if "I" is the incident flux then,

$I\sigma(v,'v:v^*,'v^*)d\Omega \equiv$ number of particles deflected per sec. into the solid angle $d\Omega$ (see figure 3) centered

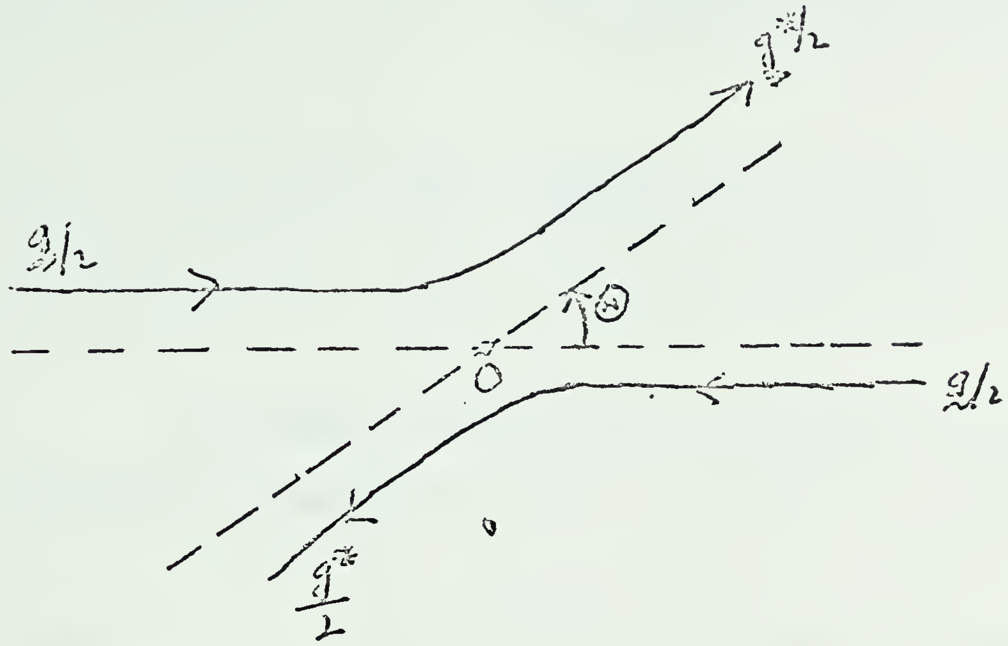


FIGURE 2.

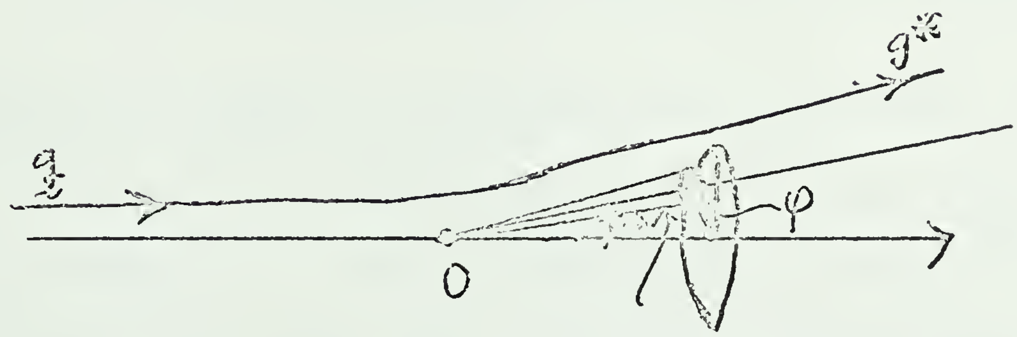


FIGURE 3

about the angle Θ between $\underset{\sim}{g}$ and $\underset{\sim}{g}^*$ in the direction of $\underset{\sim}{g}^*$.

$\sigma(v,'v:v^*,'v^*)$ signifies collisions of the type where particles with velocities $\underset{\sim}{v}$, and $'\underset{\sim}{v}$, go into velocities v^* , and $'v^*$.

The number of collisions occuring in d^3r during the interval dt of the type where $\underset{\sim}{v}$, and $'\underset{\sim}{v}$ go into v^*, and $'v^*$ is

$$(2\text{-}10) \qquad N(\underset{\sim}{r},'\underset{\sim}{v},t)d^3{'v}\ |\underset{\sim}{g}|\ \sigma(v,'v:v^*'v^*)\ d\Omega\ dt \quad .$$

The solid angle $d\Omega$ is a compact way of denoting the velocity ranges d^3v^* and $d^3{'v}^*$ about v^* , and $'v^*$ respectively. Multiplying (2-10) by the number of scattering centers, (2-9), we get

$$(2\text{-}11) \qquad \sigma(v,'v:v^*,'v^*)\ |\underset{\sim}{g}|\ N(\underset{\sim}{r},\underset{\sim}{v},t)\ N(\underset{\sim}{r},'v,t)\ d^3v\ d^3{'v}\ d\Omega\ dt \quad .$$

This represents the number of collisions that occur in the time interval dt of the type where the incident particles are in the range d^3v and $d^3{'v}$ centered about $\underset{\sim}{v}$, and $'\underset{\sim}{v}$ respectively, and where the final particles are in the ranges d^3v^* and $d^3{'v}^*$ centered about $\underset{\sim}{v}^*$, and $'v^*$ respectively. Since we are interested only in those particles with velocity $\underset{\sim}{v}$ that suffer collisions, the other incident particles $'\underset{\sim}{v}$ can have any value, and the scattered particles can go into any value. Thus the number of collisions per unit time occuring to particles in the range d^3v is

$$(2\text{-}12) \quad J_{loss} d^3v \equiv d^3v \int d^3{}'v \int d\Omega \; \sigma(v,'v:v^*,'v^*)|\underset{\sim}{g}| \; N(\underset{\sim}{r},\underset{\sim}{v},t) \; N(\underset{\sim}{r},'\underset{\sim}{v},t) \; .$$

Equation 2-12 then tells us the number of particles per unit time that leave our elemental volume d^3v due to collisions.

To calculate the number of particles that enter our elemental volume per unit time, we perform similar arguments to the above, where we now seek collisions of the type where the initial velocities can have any value and one of the scattered particles can have any velocity range, but where the other scattered particle has a range d^3v about $\underset{\sim}{v}$. Therefore the number of particles that enter the range d^3v about $\underset{\sim}{v}$ per unit time is

$$(2\text{-}13) \quad J_{gain} d^3v \equiv d^3v^* \int d^3{}'v^* \int d\Omega \; \sigma(v^*,'v^*:v,'v)|\underset{\sim}{g}^*| \; N(\underset{\sim}{r},v^*,t)N(\underset{\sim}{r},'v^*,t) \; .$$

From equations 2-12, 2-13, it follows that

$$(2\text{-}14) \qquad (J_{gain} - J_{loss})d^3v \equiv \iint \sigma(v^*,'v^*:v,'v) \cdot |\overset{*}{\underset{\sim}{g}}| \quad N(\underset{\sim}{r},\underset{\sim}{v}^*,t) \; \times$$

$$\times \; N(\underset{\sim}{r},'\underset{\sim}{v},t)d^3v^*d^3{}'v^* d\Omega \; - \iint \sigma(v,'v:v^*,'v^*) \; |\underset{\sim}{g}| \; \times$$

$$\times \; N(\underset{\sim}{r},\underset{\sim}{v},t) \; N(r,'v,t)d^3v d^3{}'v \; d\Omega \qquad .$$

We will now assume microscopic reversibility in collisions so that

$$\sigma(v^*,'v^*:v,'v) = \sigma(v,'v:v^*,'v^*) \quad . \tag{2-15}$$

Making use of equations 2-7, 2-15, and the fact that $d^3v\, d^3{}'v = d^3v^*\, d^3{}'v^*$ (1) , equation 2-14 reduces to

$$(J_{gain} - J_{loss})d^3v = d^3v \iint \sigma(v,'v:v^*,'v^*)\ |\underset{\sim}{g}|\ \{N(\underset{\sim}{r},\underset{\sim}{v}^*,t) \times$$

$$\times\, N(\underset{\sim}{r},'\underset{\sim}{v}^*,t) - N(\underset{\sim}{r},'\underset{\sim}{v},t)N(\underset{\sim}{r},\underset{\sim}{v},t)\}d^3{}'v\, d\Omega \quad . \tag{2-16}$$

Using equation 2-16, equation 2-3 finally yields the classical Boltzmann transport equation, i.e.,

$$\frac{\partial N}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial N}{\partial \underset{\sim}{r}} + \frac{1}{m}\underset{\sim}{F} \cdot \frac{\partial N}{\partial \underset{\sim}{v}} = \iint \sigma(v,'v:v^*,'v^*)|\underset{\sim}{g}|\ (N^*\,'N^* - 'NN)d^3{}'v\, d\Omega \tag{2-17}$$

where $\sigma(v,'v:v^*,'v^*)$ corresponds to the differential cross section for a collision of the type where $\underset{\sim}{v}$, and $'\underset{\sim}{v}$ go into $\underset{\sim}{v}^*$, and $'v^*$, and $N \equiv N(\underset{\sim}{r},\underset{\sim}{v},t)$, $'N \equiv N(\underset{\sim}{r},'\underset{\sim}{v},t)$, etc.

Turning to the Relativistic case, we will assume that these are no external forces except for a self-consistent background field which will take into account gravitation. To treat the case of no collisions in the relativistic case, consider an event x , and a 4-momentum P^μ at x . Also at x consider some time-like unit vector n_μ and orthogonal to it can elemental 3-flat ds . ds will serve as our target. Consider another event x' such that the 4-vector from x to x' is in the direction of P^μ , and at x' consider another elemental 3-flat of the same dimensions

(size) and orthogonal to n_μ (see figure 4). Choosing some momentum range $d\omega$ about P^μ on the pseudosphere, we can write,

$$\nu = N(x,P)\ ds\ |P^\mu n_\mu|\ d\omega$$
$$\nu' = N(x',P)\ ds\ |P^\mu n_\mu|\ d\omega \quad ,$$

which implies that

$$\frac{\nu'}{\nu} = \frac{N(x',P)}{N(x,P)} \quad . \tag{2-18}$$

Therefore the ratio, equation 2-18, is independent of the size of $d\omega$. The elementary solid angle $d\omega$ in the direction P^μ is a measure of how much the world lines fan out. For example if $d\omega = 0$, this would imply that they would not fan out about P^μ, (see figure 5).

Suppose we pick $d\omega = 0$, then the world lines will not spread, and if we don't allow any collisions the world lines that cross ds at χ will also cross ds at χ', (figure 4). This tells us that at and χ' we have the same number of particles, i.e.

$$\frac{\nu'}{\nu} = 1 \quad .$$

So that

$$\frac{N(x',P)}{N(x,P)} = 1 \quad .$$

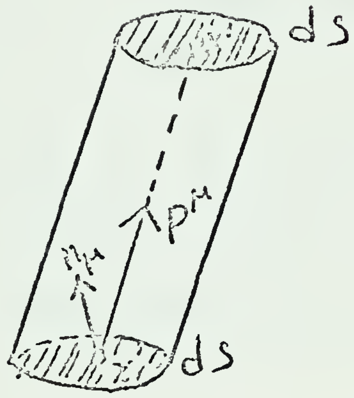


FIGURE 4

FIGURE 5

But since we know that the above ratio (equation 2-19) is independent of our choice of $d\omega$ then we can write

$$(2\text{-}20) \qquad N(\chi,P) = N(\chi',P) \quad ,$$

for all choices of "$d\omega$" . If we now let $x' \to x$ along the direction of P^μ , then the condition equation 2-20 tells us that the directional derivative of N in the direction of P^μ is zero, i.e.

$$(2\text{-}21) \qquad P^\mu \frac{\partial N}{\partial x^\mu} = 0 \quad .$$

Equation 2-21 tells us that if an observer moves along with the particles, he sees no change in the density function. We see that equation 2-21 is equivalent to the classical expression equation 2-2, if in equation 2-2 the external forces vanish.

But just as we saw earlier, this only works if there are no collisions, because otherwise world lines can leak out, and into our volume (see figure 4). To account for collisions we proceed in the same fashion as for the classical case, by replacing the right hand side of equation 2-21 with the phenomenological collision term $J_{gain} - J_{loss}$, i.e. for collisions we have

$$(2\text{-}22) \qquad P \frac{\partial N}{\partial x^\mu} = J_{gains} - J_{loss} \quad .$$

The calculation of $J_{gain} - J_{loss}$ proceeds in a fashion similar to the way we derived, equation 2-14. That is, we first have the conservation of 4-momentum

$$P_\mu + {}'P_\mu = P_\mu^* + {}'P_\mu^* \,, \tag{2-23}$$

where we are using normalized 4-momenta, (see page 13), i.e. $g_{\mu\nu} P^\mu P^\nu = -1$. The relative 4-momenta are

$$g_\mu = {}'P_\mu - P_\mu \,, \quad g_\mu^* = {}'P_\mu^* - P_\mu^* \,. \tag{2-24}$$

Note that these are spacelike vectors.

In a collision we have the condition,

$$g \equiv |\underset{\sim}{g}| = |\underset{\sim}{g}^*| = g^* \,, \quad \text{i.e.} \quad g_\mu g^\mu = g_\mu^* g^{*\mu} \tag{2-25}$$

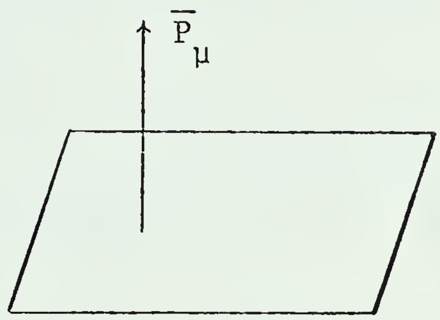
which follows simply from equations 2-23, 2-24, and the normalization condition $P_\mu P^\mu = -1$. If we define the unit vector

$$\overline{P}_\mu \equiv \frac{P_\mu + {}'P_\mu}{\sqrt{4+g^2}} \,, \quad \overline{P}_\mu^* \equiv \frac{P^*_\mu + {}'P^*_\mu}{\sqrt{4+g^{*2}}} \,. \tag{2-26}$$

We see that by equation 2-23

$$\overline{P}_\mu = \overline{P}_\mu^* \,. \tag{2-27}$$

$\overline{P}_\mu$ describes the motion of the center of mass, i.e. its tangent to the world line of the centroid and its easily seen that g_μ and g^*_μ are orthogonal to $\overline{P}_\mu$. This means that g_μ and g^*_μ lie in a 3-flat normal to $\overline{P}_\mu$, in momentum space. The 3-flat in question is called the centroidal 3-flat, i.e.



centroidal 3-flat

We translate to the center of mass frame by choosing our time axis to be along $\overline{P}_\mu$.

In the calculation of $J_{gain} - J_{loss}$ we will define a "scattering amplitude" which we will relate to the differential cross section. We define the quantity in question, i.e. W , by letting

$$W(P,'P:P^*,'P^*)\ N(x,P)\ N(x,'P)\ d\omega\ d'\omega\ d\omega^*\ d'\omega^*\ d\tau \quad , \tag{2-28}$$

be the number of binary collisions occuring in the 4-volume $d\tau$ centered about x^μ , where the initial particles are in ranges $d\omega$, and $d'\omega$ about P_μ , and $'P_\mu$ respectively, and where the final momenta are in ranges $d\omega^*$, and $d'\omega^*$, about P^*_μ , and $'P^*_\mu$ respectively. One should note that this is the relativistic expression analogous to (2-11),

in the classical case. To calculate J_{loss} , we use the same arguments as before. That is, we want to know how many particles in the range $d\omega$ about P^{μ} , leave due to collisions occuring in $d\tau$. To do this we consider the number of collisions that occur in $d\tau$ with one of the initial particles in the range $d\omega$ about P_{μ} , the other initial particle with arbitrary value, and the scattered particles with arbitrary value. This number is

$$(2\text{-}29) \qquad N(x,P)d\omega\, d\tau \iiint W(P,'P:P^{*},'P^{*})\, N(x,'P)\, d'\omega\, d\omega^{*}\, d'\omega^{*} \quad .$$

To calculate J_{gain} we are interested in those particles which after collision, enter the range $d\omega$ about P_{μ} . We calculate this gain by considering the inverse collision, i.e. we assume that the initial particles can have any values, as well as one of the scattered particles, but that the other scattered particle must go into the range $d\omega$. The "scattering amplitude" for this is $W(P^{*},'P^{*}:P,'P)$. Therefore by equation 2-28 we get that the number of collisions satisfying the above condition must be

$$(2\text{-}30) \qquad d\omega\, d\tau \iiint W(P^{*},'P^{*}:P,'P)\, N(x,P^{*})\, N(x,'P^{*})\, d\omega^{*}\, d'\omega^{*}\, d'\omega \quad .$$

Since equation 2-22 holds per unit volume in phase space (2) , (in our case this means "per unit volume per unit time per unit volume in momentum space"), therefore

$$(2\text{-}31) \qquad (J_{gain}-J_{loss})\,d\omega d\tau = d\omega d\tau \iiint W(P^*,'P^*:P,'P)N^*\,'N^*\,d\omega^*\,d'\omega^*\,d'\omega$$

$$- \, d\omega d\tau \iiint W(P,'P:P^*,'P^*)\; N'N\, d\omega^*\, d'\omega^*\, d'\omega \;.$$

Making use of the symmetry $W(P,'P:P^*,'P^*) = W(P^*,'P^*:P,P^*)$, i.e. microscopic reversibility, equation 2-31, equation 2-22 finally becomes

$$(2\text{-}32) \qquad P^\mu \frac{\partial N}{\partial x^\mu} = \iiint W(P,'P:P^*,'P^*)\;(N^*\,'N^*-N'N)\; d'\omega\; d\omega^*\; d'\omega^* \quad .$$

Equation 2-32 is Relativistic Boltzmann Collision equation in the absence of external forces (i.e. of a non-gravitational nature).

Next we will show that equation 2-32 is analogous to the classical case, i.e. equation 2-17. We will assume that the collision process has axial symmetry about the line of centers of the colliding particles. We define the angle Θ between g_μ^* and g_μ by the relation,

$$(2\text{-}33) \qquad \text{Cos}\;\Theta \equiv \frac{g_\mu^*\; g_\mu}{g^2} \quad .$$

Since the collision process in question is elastic the condition (2-25), and (2-27) must be satisfied. We also know that just as in the classical case, an elastic collision is determined by g and Θ . Incorporating all this, we can write

$$(2\text{-}34) \qquad W(P,'P:P^*,'P^*) = W_1(g,\Theta)\;\delta_\omega(\bar{P}_\mu^* - \bar{P}_\mu)\;\delta(g^*-g) \quad .$$

The scattering cross section $\sigma(g,\Theta)$ is introduced by writing

$$(2\text{-}35) \qquad W(P,{}'P{:}P^*,{}'P^*) = (1 + \tfrac{1}{4} g^2)^{-\frac{1}{2}} g^{-1} \sigma(g,\Theta)\, \delta_\omega(\overline{P}_\mu^{\,*} - \overline{P}_\mu)\,\delta(g^* - g) .$$

The reason for calling the quantity $\sigma(g,\Theta)$ the cross section (3) is that, by substituting equation 2-35 into equation 2-28, we get

$$(2\text{-}36) \qquad \sigma(g,\Theta) N'N\, d\omega\, d'\omega \frac{1}{(1 + \frac{g^2}{4})^{\frac{1}{2}}} \frac{1}{g} \delta_\omega(\overline{P}_\mu^{\,*} - \overline{P}_\mu)\, \delta(g^* - g)\, d\omega^*\, d'\omega^*\, d\tau .$$

We also know that in an elastic collision satisfying $P_\mu + {}'P_\mu = P_\mu^* + {}'P_\mu^*$ we can describe the initial momenta relative to some fixed tetrad (M_1, M_2, M_3, M_4) in momentum space by either:

(1) specifying χ, θ, ϕ of P_μ and ${}'\chi, {}'\theta, {}'\phi$ of ${}'P_\mu$ (see equation 1-18), as shown in figure 6.

(2) specifying $\overline{\chi}, \overline{\theta}, \overline{\phi}$ of $\overline{P}_\mu$, and then, in the centroidal 3-flat orthogonal to $\overline{P}_\mu$, having some fixed orthonormal triad from which the direction of g_μ is specified by ϑ, and ϵ, as well as of course, g itself, (see figure 7).

A relationship among the elemental 3-areas $d\omega$, $d'\omega$ and $\overline{d\omega}$ is (4)

$$(2\text{-}37) \qquad d\omega\, d'\omega = (1 + \tfrac{1}{4} g^2)^{\frac{1}{2}}\, \overline{d\omega}\, g^2\, dg\, \sin\vartheta\, d\vartheta\, d\epsilon .$$

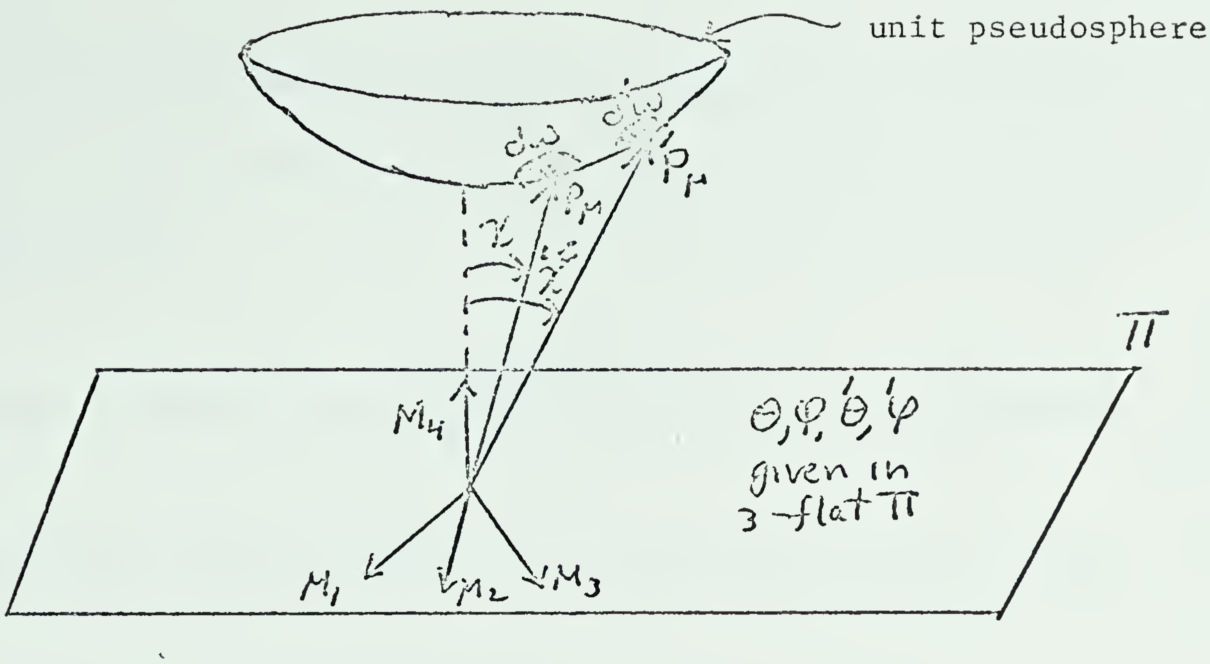


FIGURE 6

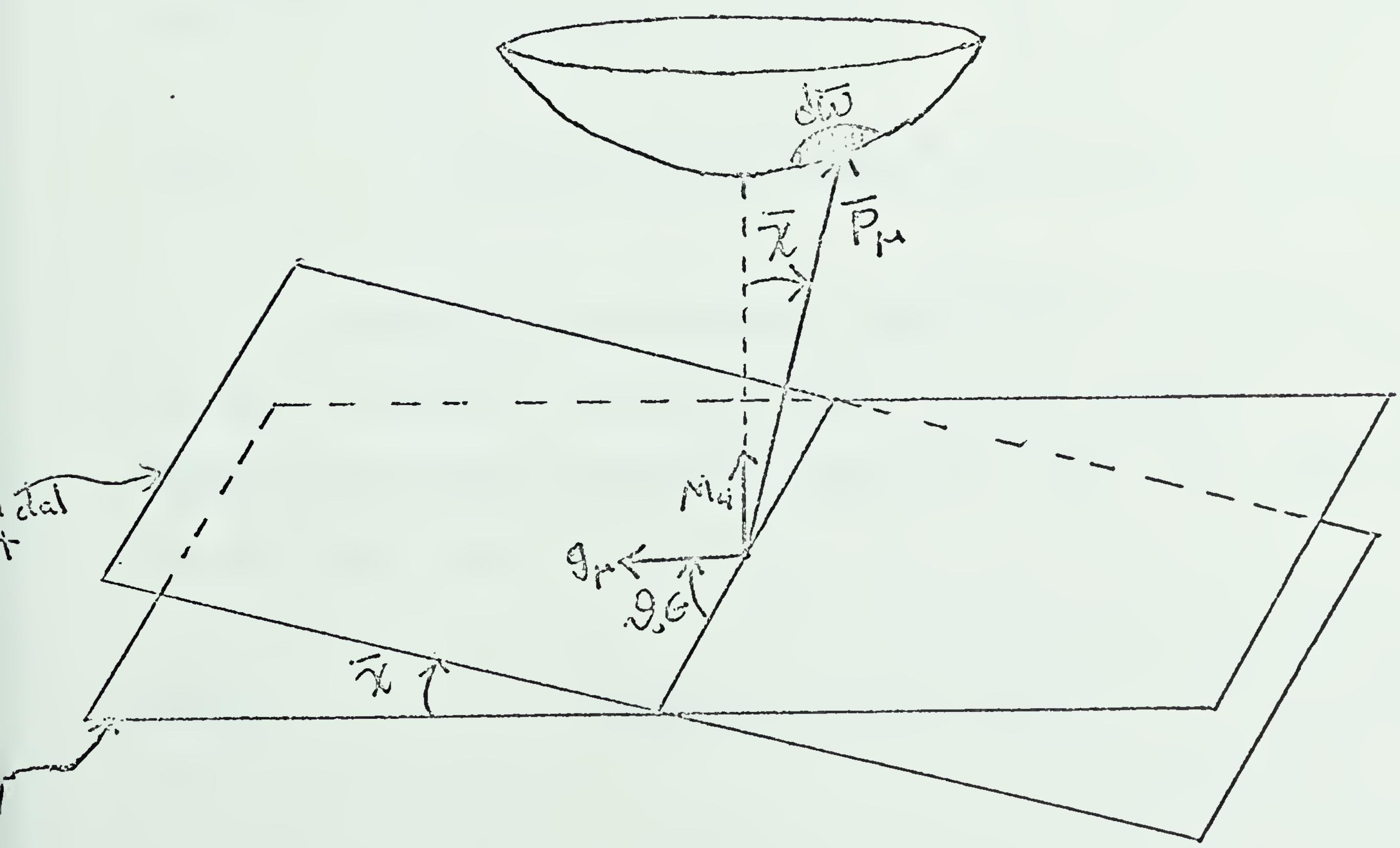


FIGURE 7

We can describe the final momenta in exactly the same fashion as was done for the initial momenta. For the final momenta we also have the relation (4)

$$(2\text{-}38) \qquad d\omega^* \, d'\omega^* = (1 + \frac{1}{4} g^{*2})^{\frac{1}{2}} \, d\overline{\omega}^* \, g^{*2} \, dg^* \, \sin\vartheta^* \, d\vartheta^* \, d\epsilon^* \quad .$$

Subsituting equation 2-38 into the expression 2-36 we have,

$$(2\text{-}39) \qquad N'N \, \sigma(g,\Theta) \frac{d\omega \, d'\omega}{g(1 + \frac{1}{4} g^2)^{\frac{1}{2}}} \, g^{*2}(1 + \frac{1}{4} g^{*2}) \, \delta(g^* - g) \, dg^* \times$$

$$\times \, \delta\omega(\overline{P}_\mu^* - \overline{P}_\mu) d\overline{\omega}^* \, \sin\vartheta^* \, d\vartheta^* \, d\epsilon^* \, d\tau \quad .$$

It is seen that the only "time" (2-38) is non-vanishing is on the unit pseudosphere when $\overline{P}_\mu^* = \overline{P}_\mu$, and when $g = g^*$. 2-39 thereby becomes

$$(2\text{-}40) \qquad g\sigma(g,\Theta) \, N'N \, d\omega \, d'\omega \, \sin\vartheta^* \, d\vartheta^* \, d\epsilon^* \, d\tau \quad .$$

Replacing the solid angle $\sin\vartheta^* \, d\vartheta^* \, d\epsilon^*$ by $d\Omega$ we can then say that the number of collisions with relative 3-momenta g , involving particles with initial momenta in ranges $d\omega$, and $d'\omega$, that scatter into the solid angle $d\Omega$ is (5)

$$(2\text{-}41) \qquad g \, \sigma(g,\Theta) \, N'N \, d\omega \, d'\omega \, d\Omega \, d\tau \; .$$

We can thus rewrite the Boltzmann collision equation to get,

$$P^{\mu} \frac{\partial N}{\partial x^{\mu}} = \iint g\ \sigma(g,\Theta)('N^{*}N^{*} - 'NN)\ d\Omega\ d'\omega \quad . \tag{2-42}$$

Equation 2-42 is the relativistic analog of the classical Boltzmann collision equation, (2-17).

CHAPTER III

Conservation Laws

In this chapter certain general conservation laws will be shown to be direct consequences of the Boltzmann Collision Equation. We will also show how one can obtain Numerical flux vector and Energy-Momentum tensor of a Perfect Fluid with explicit representations for P, ρ, and μ.

As was mentioned earlier a non-equilibrium situation will tend to equilibrium by a reordering if you will, of non-uniformities. In so doing certain quantities will be transported, e.g. heat will be conducted in order to smooth out temperature differences, momentum flows in order to smooth out pressure differences, "mass" flows in order to smooth out density differences, etc. These flows or transport effects can be described once we have solved the Boltzmann collision equation for the distribution function N. But there are certain facts which can be deduced about these flows without explicitly solving the Boltzmann equation. These facts take the form of conservation laws. Let us first consider the classical case.

Consider a binary collision taking place at some point $\underset{\sim}{r}$. Let $\psi(\underset{\sim}{r}, \underset{\sim}{v})$ be any conserved quantity associated with a particle whose velocity is $\underset{\sim}{v}$ i.e.,

(3-1) $$\psi + {}'\psi = \psi^* + {}'\psi^* \quad .$$

Consider the following expression

(3-2) $$J(\psi) \equiv \iiint \sigma(v,{}'v:v^*,{}'v^*)\,|\underset{\sim}{g}|\,\psi({}'N^*N^* - {}'NN)\,d\Omega\, d^3v\, d^3{}'v \quad .$$

$J(\psi)$ is just the collision term in the Boltzmann equation multiplied by ψ and integrated over $\underset{\sim}{v}$. We now write three equivalent expressions of the above integral by performing a series of transformations of the variables of integration, and making use of some symmetries of σ . Firstly, we let $v \to {}'v$, and ${}'v \to v$, then equation 3-2 becomes

(3-3) $$J({}'\psi) = \iiint \sigma({}'v,v:v^*{}'v^*)\,|\underset{\sim}{g}|\,{}'\psi({}'N^*N^* - {}'NN)\, d\Omega\, d^3v\, d^3{}'v \quad .$$

By means of rather obvious symmetry property, i.e.

$$\sigma(v,{}'v:v^*,{}'v^*) = \sigma({}'v,v:v^*{}'v^*) \quad ,$$

equation 3-3 becomes

(3-4) $$J({}'\psi) = \iiint \sigma(v{}'v:v^*,{}'v^*)\,|\underset{\sim}{g}|\,{}'\psi({}'N^*N^* - {}'NN)\, d\Omega\, d^3v\, d^3{}'v \, .$$

By next performing the following relabeling of variables,

$$v \to v^* \qquad v^* \to v$$
$${}'v \to {}'v^* \quad {}'v^* \to {}'v$$

from equation 4-2 we get

$$(3\text{-}5) \qquad J(\psi^*) = \iiint \sigma('v^*,v^*:v,'v)\ \psi^*(N'N - N^*{}'N^*)d\Omega\ d^3v^*\ d^3{}'v^*$$

$$= -\iiint \sigma(v,'v:v^*{}'v^*)\ \psi^*('N^*N^* - N'N)\ d\Omega\ d^3v\ d^3{}'v\ ,$$

where we have made use of microscopic reversibility, Liouville's theorem (i.e. $d^3v\ d^3{}'v = d^3v^*\ d^3{}'v^*$) , and equation 2-7. By similar arguments it can be shown that

$$(3\text{-}6) \qquad J('\psi^*) = -\iiint \sigma(v,'v:v^*,'v^*)\ '\psi^*('N^*N^*-N'N)\ d\Omega\ d^3v\ d^3{}'v\ .$$

Since equations 3-2,4,5,6 are all equivalent expressions then clearly

$$(3\text{-}7) \qquad J(\psi) = \frac{1}{4}\ (J(\psi) + J('\psi) + J('\psi^*) + J(\psi^*))\ .$$

Substituting equations 3-2,4,5,6 into equation 3-7 we get,

$$(3\text{-}8) \qquad J(\psi) = \frac{1}{4} \iiint \sigma(v,'v:v^*,'v^*)('N^*N^*-'NN)(\psi+'\psi-\psi^*-'\psi^*)\ d\Omega\ d^3v\ d^3{}'v\ .$$

But ψ is a collisional invariant, so that $J(\psi) \equiv 0$. It then follows from equation 3-2, and equation 2-17, that

$$(3\text{-}9) \qquad \int \psi(\frac{\partial}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial}{\partial \underset{\sim}{r}} + \frac{1}{m} \underset{\sim}{F} \cdot \frac{\partial}{\partial \underset{\sim}{v}})\ N(\underset{\sim}{r},\underset{\sim}{v},t)\ d^3v = 0\ .$$

Equation 3-9 contains the crux of our conservation laws and transport effects. Writing equation 3-9 out explicitly and then adding and subtracting similar terms we have

$$
\begin{aligned}
(3\text{-}10) \qquad 0 &= \int \psi \frac{\partial N}{\partial t} d^3v + \int \psi\, v_i \frac{\partial N}{\partial x_i} + \frac{1}{m}\int \psi F_i \frac{\partial N}{\partial v_i} d^3v \qquad i = 1,2,3 \\
&= \frac{\partial}{\partial t}\int \psi N\, d^3v + \int \psi\, v_i \frac{\partial N}{\partial x_i} d^3v + \int v_i \frac{\partial \psi}{\partial x_i} N\, d^3v - \int v_i \frac{\partial \psi}{\partial x_i} N\, d^3v \\
&+ \frac{1}{m}\int \psi F_i \frac{\partial N}{\partial x_i} d^3v + \frac{1}{m}\int \psi \frac{\partial F_i}{\partial x_i} N\, d^3v - \frac{1}{m}\int \psi \frac{\partial F_i}{\partial x_i} N\, d^3v \\
&+ \frac{1}{m}\int F_i \frac{\partial \psi}{\partial v_i} N\, d^3v - \frac{1}{m}\int F_i \frac{\partial \psi}{\partial v_i} N\, d^3v \ .
\end{aligned}
$$

Collecting terms equation 3-10 becomes

$$
\begin{aligned}
(3\text{-}11) \qquad & \frac{\partial}{\partial t}\int \psi N\, d^3v + \int \frac{\partial}{\partial x_i}(\psi v_i N)\, d^3v - \int \frac{\partial \psi}{\partial x_i} v_i N\, d^3v \quad + \\
& + \frac{1}{m}\int \frac{\partial}{\partial v_i}(\psi F_i N)\, d^3v - \frac{1}{m}\int \frac{\partial \psi}{\partial v_i} F_i\, d^3v \; - \\
& - \frac{1}{m}\int \psi \frac{\partial F_i}{\partial v_i} N\, d^3v = 0 \ .
\end{aligned}
$$

By Gauss's theorem the fourth term becomes

$$
\frac{1}{m}\int \frac{\partial}{\partial v_i}(\psi F_i N)\, d^3v = \frac{1}{m}\int_{v\to\infty} ds\ \underset{\sim}{n}\cdot(\psi \underset{\sim}{F} N) \ .
$$

But because of the fact that $N \to 0$ as $v \to \infty$ we can neglect this term. Defining the mean with respect to the distribution function N, of a

quantity A to be,

$$\overline{A} \equiv \frac{\int A \; N \; d^3v}{\int N \; d^3v} = \frac{1}{n} \int A \; N \; d^3v \quad . \tag{3-12}$$

It must be pointed out that $\overline{A}$ is really a local mean, i.e. its an average over velocity space but it varies from point to point in position space. Using equation 3-12, equation 3-11 becomes

$$\frac{\partial}{\partial t} \overline{(n\psi)} + \frac{\partial}{\partial x_i} \overline{(n\psi v_i)} - n \overline{(v_i \frac{\partial \psi}{\partial x_i})} - \frac{n}{m} \overline{(F_i \frac{\partial \psi}{\partial v_i})} - \frac{n}{m} \overline{(\frac{\partial F_i}{\partial v_i} \psi)} = 0 \; . \tag{3-13}$$

Let us for the moment consider the situation where $\underset{\sim}{F}$ is independent of velocity, equation 3-13 then becomes

$$\frac{\partial}{\partial t} \overline{(n\psi)} + \frac{\partial}{\partial x_i} \overline{(n\psi v_i)} - n \overline{(v_i \frac{\partial \psi}{\partial x_i})} - \frac{n}{m} \overline{(F_i \frac{\partial \psi}{\partial v_i})} = 0 \quad . \tag{3-14}$$

The collisional invariants we have are:

(1) ψ = constant, i.e. $\psi = 1$ (for convenience).

(2) conservation of mass, i.e. $m + {'m} = m^* + {'m^*}$.

(3) conservation of momentum, i.e. $\psi = mv_i$.

(4) conservation of thermal energy, i.e. $\psi = \frac{1}{2} m \, |\underset{\sim}{v} - \overline{\underset{\sim}{v}}|^2$,

where we call $\underset{\sim}{v} - \overline{\underset{\sim}{v}}$ the thermal velocity.

The four collisional invariants, by means of equation 3-14 give rise to 4 conservation laws. For $\psi = 1$ we get the continuity equation for n , i.e. $\frac{\partial n}{\partial t} + \underset{\sim}{\nabla} \cdot (n\underset{\sim}{v}) = 0$.

For $\psi = m$ it yields the continuity equation i.e. 3-14 becomes,

$$\frac{\partial}{\partial t}\overline{(mn)} + \frac{\partial}{\partial x_i}\overline{(nv_i m)} - n\overline{(v_i \frac{\partial m}{\partial x_i})} - \frac{n}{m}\overline{(F_i \frac{\partial m}{\partial v_i})} = 0$$

which reduces to

(3-15) $$\frac{\partial \overline{(nm)}}{\partial t} + \frac{\partial}{\partial x_i}\overline{(nm\ v_i)} = 0 \ .$$

Defining density of mass as $\rho \equiv mn$, equation 3-15 becomes

(3-16) $$\frac{\partial \rho}{\partial t} + \underset{\sim}{\nabla}\cdot(\rho\overline{\underset{\sim}{v}}) = 0 \quad , \quad \text{continuity equation.}$$

Putting $\psi = mv$, equation 3-14 becomes

(3-17) $$\frac{\partial}{\partial t}\overline{(\rho v_j)} + \frac{\partial}{\partial x_i}\overline{(\rho v_i v_j)} - \frac{\rho}{m}\overline{(F_i\ \delta_{ij})} = 0 \quad .$$

To get equation 3-17 into a more familiar form we can use the fact that

(3-18) $$\overline{(v_i v_j)} = \overline{(v_i - \overline{v}_i)(v_j - \overline{v}_j)} + \overline{v}_i\overline{v}_j \quad .$$

Defining a Pressure Tensor P_{ij} to be

$$P_{ij} \equiv \rho\overline{(v_i - \overline{v}_i)(v_j - \overline{v}_j)} \quad ;$$

and making use of equation 3-18, 19, equation 3-17 becomes

$$\left(\frac{\partial}{\partial t} + \bar{v}_i \frac{\partial}{\partial x_i}\right) \bar{v}_j = \frac{1}{m} F_j - \frac{1}{\rho} \frac{\partial}{\partial x_i} P_{ij} \quad , \qquad \text{or}$$

(3-20) $$\rho\left(\frac{\partial}{\partial t} + \bar{\underset{\sim}{v}}\cdot\nabla\right)\bar{\underset{\sim}{v}} = \frac{\rho}{m} \underset{\sim}{F} - \underset{\sim}{\nabla} \cdot \overleftrightarrow{P} \quad ,$$

which is just another form of Newton's 2nd law.

Finally for $\psi = \frac{1}{\alpha} m \, |\underset{\sim}{v} - \bar{\underset{\sim}{v}}|^2$, equation 3-14 becomes

(3-21) $$\frac{1}{2} \frac{\partial}{\partial t} \overline{(\rho|\underset{\sim}{v}-\bar{\underset{\sim}{v}}|^2} + \frac{\partial}{\partial x_i} \overline{(\rho v_i |\underset{\sim}{v}-\bar{\underset{\sim}{v}}|^2} - \frac{1}{2} \rho \, \overline{(v_i \frac{\partial}{\partial x_i} |\underset{\sim}{v}-\bar{\underset{\sim}{v}}|^2)} = 0 \quad .$$

Defining temperature by,

(3-22) $$KT \equiv \Theta \equiv \frac{1}{2} m \, \overline{(|\underset{\sim}{v}-\bar{\underset{\sim}{v}}|^2)} \quad ,$$

and heat flux by

(3-23) $$\underset{\sim}{q} \equiv \frac{1}{2} m \rho \, \overline{((v-\bar{v})|v-\bar{v}|^2)} \quad ,$$

equation 3-21 reduces to

(3-24) $$\frac{3}{2} \frac{\partial}{\partial t} (\rho\Theta) + \frac{\partial q_i}{\partial x_i} + \frac{3}{2} \frac{\partial}{\partial x_i} (\rho\Theta\bar{v}_i) + m \, P_{ij} \frac{\partial \bar{v}_i}{\partial x_i} = 0 \quad ,$$

where we have used that fact that $\rho \, \overline{(v_i(v_j-\bar{v}_j))} = P_{ij}$. Noting that P_{ij} is symmetric, equation 3-24 can be written as,

(3-25) $$\frac{3}{2}\frac{\partial}{\partial t}(\rho\Theta) + \frac{\partial q_i}{\partial x_i} + \frac{3}{2}\frac{\partial}{\partial x_i}(\rho\Theta\bar{v}_i) + \frac{m}{2}P_{ij}\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial \bar{v}_i}{\partial x_j}\right) = 0 .$$

Defining $\Lambda_{ij} \equiv \frac{1}{2} m \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$, equation 3-25 becomes

(3-26) $$\rho\left(\frac{\partial}{\partial t} + \bar{\underset{\sim}{v}}\cdot\nabla\right)\Theta = -\frac{2}{3}\underset{\sim}{\nabla}\cdot\underset{\sim}{q} - \frac{2}{3}\overleftrightarrow{P}\cdot\overleftrightarrow{\Lambda} .$$

At this point we should note that the preceeding ideas of how to relate heat flow, temperature, etc. to microscopic variables must be compared with experiment. But the comparison with experiment can't be made until we have some notion as to what the distribution function should be.

The calculation of the conservation laws becomes much simpler in the relativistic case. We start by defining

(3-27) $$J(\psi) \equiv \iiiint W(P,'P:P^*,'P^*)\psi(x,P)('N^*N^*-'NN)d'\omega^*d\omega^*d'\omega\, d\omega$$

where $\psi(x,P)$ is any tensor function of x^μ , and P_ν . Then in exactly the same fashion as equation 3-8 was derived, we can prove that

(3-28) $$J(\psi) = \frac{1}{4}\iiiint W(P,'P:P^*,'P^*)[\psi+'\psi-\psi^*-'\psi^*]('N^*N^*-'NN)d'\omega^*d\omega^*d'\omega d\omega .$$

In the relativistic case there are only two collision invariants:

(3-29) $$\psi = 1 \quad , \quad \psi = P^{\mu} \quad .$$

The apparent simplicity in the number of collision invariants is essentailly due to our space-time approach. In the classical case we had momentum, and "thermal energy" as collisional invariants. But now the 4-momentum vector P^{μ} incorporates both these conditions. The other collisional invariant of mass is taken into account by the restriction that all the 4-momenta vectors must terminate on the same fixed pseudosphere.

From equation 3-28 we see that when $\psi = 1$ or P^{μ} we have,

(3-30) $$J(\psi) = 0 \quad .$$

Thus in view of equation 3-30 we see that equation 2-32 leads to the result

(3-31) $$\int \psi \, P^{\nu} \frac{\partial N}{\partial x^{\nu}} \, d\omega = 0 \quad .$$

Equation 3-31 is the relativistic analog of equation 3-9.

For $\psi = 1$ equation 3-31 yields,

$$\int P^{\nu} \frac{\partial N}{\partial x^{\nu}} \, d\omega = 0 \quad ,$$

or,

(3-32) $$\frac{\partial}{\partial x^{\nu}} \int P^{\nu} N \, d\omega = 0 \quad .$$

For $\psi = P^{\mu}$ equation 3-31 yields

$$\int P^{\mu} P^{\nu} \frac{\partial N}{\partial x^{\nu}} \, d\omega = 0 \quad ,$$

or,

(3-33) $$\frac{\partial}{\partial x^{\nu}} \int P^{\mu} P^{\nu} N \, d\omega = 0 \quad .$$

Noting that the Energy-Momentum tensor is,

$$T^{\mu\nu} = m \, c^{2} \int P^{\mu} P^{\nu} N \, d\omega \quad ,$$

and the Numerical flux vector is,

$$M^{\mu} = m \, c \int P^{\mu} N \, d\omega \quad ,$$

we see that in general equations 3-32, 33 become

(3-34) $$M^{\mu}{}_{|\mu} = 0$$

(3-35) $$T^{\mu\nu}{}_{|\nu} = 0 \quad .$$

Equations 3-34, 35 become our conservation laws.

Perfect Fluid

As an application of the preceeding ideas the Numerical flux vector, and the Energy-Momentum tensor of a perfect fluid will be derived, with the explicit expressions for the pressure, energy density, and density of proper mass.

Before we do this it will be necessary to derive the equilibrium distribution function. The relativistic derivation of the equilibrium distribution is similar to that of the classical case. That is, we first formulate the relativistic H-theorem and then use the equilibrium condition to obtain the equilibrium distribution.

We define the entropy-flux vector to be

$$S^{\mu} \equiv - K \int N(x,P) \log N(x,P) P^{\mu} d\omega \tag{3-36}$$

where K is the Boltzmann constant. Suppose that in equation 3-28 we put $\psi = \log N$, then we get

$$\begin{aligned} J(\psi) &= \frac{1}{4} K \iiiint W(P,'P:\overset{*}{P},'\overset{*}{P})(\log N + \log 'N - \log \overset{*}{N} - \log '\overset{*}{N}) \\ &\qquad ('\overset{*}{N}\overset{*}{N} - 'NN) d'\overset{*}{\omega} d\overset{*}{\omega} d'\omega d\omega \\ &= - \frac{1}{4} K \iiiint W(P,'P:\overset{*}{P},'\overset{*}{P})(\log('\overset{*}{N}\overset{*}{N}) - \log('NN)) \\ &\qquad ('\overset{*}{N}\overset{*}{N} - 'NN) d'\overset{*}{\omega} d\overset{*}{\omega} d'\omega\, d\omega . \end{aligned} \tag{3-37}$$

But we also know that from equation 2-31, 27, we have

$$\text{(3-38)} \qquad \int \log N \, P^{\mu} \frac{\partial N}{\partial x^{\mu}} \, d\omega = J(\psi) \quad .$$

It can also be seen that

$$\text{(3-39)} \quad S^{\mu}{}_{|\mu} = - \frac{\partial}{\partial x^{\mu}} \int N \log N \, P^{\mu} \, d\omega = - \int \log N \, P^{\mu} \frac{\partial N}{\partial x^{\mu}} \, d\omega - \int P^{\mu} \frac{\partial N}{\partial x^{\mu}} \, d\omega \; .$$

By equation 3-32, we see that equation 3-39 simplies to

$$\text{(3-40)} \qquad S^{\mu}{}_{|\mu} = - \int P^{\mu} \log N \frac{\partial N}{\partial x^{\mu}} \, d\omega \quad .$$

Noting equations 3.37, and 3-38, equation 3-40 becomes

$$\text{(3-41)} \quad S^{\mu}{}_{|\mu} = \frac{1}{4} K \iiiint W(P,'P:P^{*},'P^{*}) \, [\log('N^{*}N^{*}) - \log('NN)] \times$$

$$\times \, ['N^{*}N^{*} - 'NN] d'\omega^{*} \, d\omega^{*} \, d'\omega \, d\omega \quad .$$

In equation 3-41 consider the expression $[\log('N^{*}N^{*}) - \log('NN)] \times$ $['N^{*}N^{*} - 'NN]$, where we know that $N, 'N, N^{*}$, and $'N^{*} \geq 0$. This expression is of the form $(x-y) \log (\frac{x}{y})$. Consider the following 3 cases:

(1) if $x > y$ then, $(x-y) > 0 \wedge \log (\frac{x}{y}) > 0$, therefore $(x-y) \log (\frac{x}{y}) > 0$.

(2) if $x = y$ then $(x-y) \log (\frac{x}{y}) = 0$.

(3) if $x < y$ then $(x-y) < 0 \wedge \log(\frac{x}{y}) < 0$, thus $(x-y) \log(\frac{x}{y}) > 0$.

The result of this being that

$$\forall x \geq 0 \; \forall y \geq 0 \; (x-y) \log(\frac{x}{y}) \geq 0 \quad . \tag{3-42}$$

Using equation 3-42, and noting that $W(P,'P:P^*,'P^*)$ is positive definite we see that equation 3-41 yields

$$S^{\mu}{}_{|\mu} \geq 0 \quad , \tag{3-43}$$

which is Boltzmann's H-Theorem. The connection between equations 3-43, equations 1-31 and the classical H-Theorem can be seen by letting n^{μ} be coincident to P^{μ} , and then picking the time axis along P^{μ} (1).

We know that the equilibrium condition is characterized by

$$S^{\mu}{}_{|\mu} = 0 \quad . \tag{3-44}$$

From equation 3-41 we can see that

$$S^{\mu}{}_{|\mu} = 0 \quad \text{iff} \quad \log N + \log {}'N = \log N^* + \log {}'N^* \quad . \tag{3-45}$$

The equation $\log N + \log {}'N = \log N^* + \log {}'N^*$ represents some collisional invariance. But we know that the only collisional invariants are

P^μ and a constant. Therefore log N must be some linear combination of P^μ and a constant. Letting the constant be log (x) we have

$$\text{(3-46)} \qquad \log N_o(x,P) = \log\alpha(x) + \beta_\mu(x)\, P^\mu$$

where N_o denotes the equilibrium distribution. Therefore we can write

$$\text{(3-47)} \qquad N_o(x,P) = \alpha(x)\, e^{\beta_\mu(x)\, P^\mu}$$

where $\beta_\mu(x)$ must be timelike so that $\beta_\mu(x)P^\mu \leq 0$, and thus guaranteeing that $\int N_o\, d\omega$ is finite. Letting

$$\beta^2 \equiv -\beta_\mu(x)\beta^\mu(x) \quad , \quad \text{and} \quad \beta_\mu(x) \equiv \frac{1}{c}\beta\, u_\mu \quad ,$$

then $u_\mu u^\mu = -c^2$, and by substituting 3-47 into the collision term of the Botzmann equation and noting the collisional invariance of P^μ, we have

$$\text{(3-48)} \qquad P^\mu \frac{\partial N_o}{\partial x^\mu} = 0 \quad .$$

Substituting equation 3-47 into equation 3-48 we get,

$$P^\mu \frac{\partial\ \alpha}{\partial x^\mu}(\alpha(x)e^{\beta_\nu P^\nu}) = P^\mu\left(\frac{\partial\ \alpha}{\partial x^\mu} + \alpha(x)\beta_{\nu|\mu}P^\nu\right)e^{\beta_\nu P^\nu} = 0 \ .$$

From which we can write

$$P^{\mu} \frac{\partial \alpha}{\partial x^{\mu}} + \alpha(x)\, P^{\mu}P^{\nu}\, \beta_{\nu|\mu} = 0 \quad .$$

Since P^{μ} can be any timelike unit vector we have

$$\text{(3-49)} \qquad \alpha(x) \equiv \text{const.} \quad \text{and} \quad \beta_{\nu|\mu} + \beta_{\mu|\nu} = 0 \quad .$$

Thus equation 3-49 indicates that equilibrium implies the existence of a time-like Killing vector field.

We are now ready to derive the required expression for the perfect fluid. By defining the function

$$\text{(3-50)} \qquad Z_o(\alpha, \beta_{\mu}) \equiv \alpha \int e^{\beta_{\mu}P^{\mu}}\, d\omega \quad ,$$

we see that

$$\text{(3-51)} \qquad \frac{\partial Z_o}{\partial \beta_{\mu}} = \int P^{\mu}\, \alpha\, e^{\beta_{\mu}P^{\mu}}\, d\omega = \int P^{\mu}\, N_o\, d\omega \cdot$$

and

$$\text{(3-52)} \qquad \frac{\partial^2 Z_o}{\partial \beta_{\mu} \partial \beta_{\nu}} = \int P^{\mu}P^{\nu}\, \alpha\, e^{\beta_{\mu}P^{\mu}}\, d\omega = \int P^{\mu}P^{\nu}\, N_o\, d\omega \quad .$$

Noting equations 3-34, 35, equations 3-51, 52 yield

$$(3\text{-}53) \qquad M_o^{\ \mu} = mc \frac{\partial Z_o}{\partial \beta_\mu} , \; T_o^{\ \mu\nu} = mc^2 \frac{\partial^2 Z_o}{\partial \beta_\mu \partial \beta_\nu} \quad .$$

Picking polar coordinates with β^μ parallel to the axis off which χ is measured, equation 3-50 becomes

$$(3\text{-}54) \qquad Z_o(\alpha, \beta_\mu) = \alpha \int_o^\infty \int_o^\pi \int_o^{2\pi} e^{-\beta_\mu P^\mu} \sinh^2 \chi \;\; d\chi \, \sin\theta \, d\theta \, d\phi$$

$$= \alpha \int_o^\infty \int_o^\pi \int_o^{2\pi} e^{-\beta \cosh \chi} \sinh^2 \chi \;\; d\chi \;\; \sin \theta \, d\theta \, d\phi \quad .$$

Using the formula

$$(3\text{-}55) \qquad K_n(\beta) = \frac{\beta^n}{1 \cdot 3 \cdots (2n-1)} \int_o^\infty e^{-\beta \cosh \chi} \sinh^{2n} \chi \quad d\chi$$

where $K_n(\beta)$ is the Modified Bessel function of the second order; it is seen that

$$\frac{K_1(\beta)}{\beta} = \int_o^\infty e^{-\beta \cosh \chi} \sinh^2 \chi \;\; d\chi \quad .$$

Integrating over θ , and ψ , equation 3-54 becomes

$$(3\text{-}56) \qquad Z_o(\alpha, \beta_\mu) = 4\pi \; \alpha \frac{K_1(\beta)}{\beta} \quad .$$

From equation 3-53 we have

$$M_o^{\ \mu} = m \; c \frac{\partial}{\partial \beta_\mu} \quad (4\pi \; \alpha \frac{K_1(\beta)}{\beta}) \quad ,$$

noting that

$$\frac{\partial}{\partial \beta_\mu} = \frac{\partial \beta}{\partial \beta_\mu} \frac{\partial}{\partial \beta} = - \frac{1}{c} m^\mu \frac{\partial}{\partial \beta} \quad ,$$

the above expression becomes

$$M_o^{\ \mu} = 4\pi \ \alpha \ m \ c(- \frac{1}{c} u^\mu) \frac{\partial}{\partial \beta} (\frac{K_1(\beta)}{\beta}) \quad .$$

By means of the identity

$$\frac{d}{d\beta} [\beta^{-n} K_n(\beta)] = -\beta^{-n} K_{n+1}(\beta)$$

equation 3-53 finally becomes

$$M_o^{\ \mu} = 4\pi \ \alpha \ m \frac{K_2(\beta)}{\beta} \ u^\mu \quad . \tag{3-57}$$

Letting the density $\rho = 4\pi \ \alpha \ m \dfrac{K_2(\beta)}{\beta}$ we see that equation 3-57 is the usual expression for the numberical flux vector of a perfect fluid, i.e.

$$M_o^{\ \mu} = \rho \ u^\mu \quad . \tag{3-58}$$

In a similar fashion starting with,

$$T_o^{\ \mu\nu} = m \ c^2 \frac{\partial^2 Z_o}{\partial \beta_\mu \partial \beta_\nu} = 4\pi \ \alpha \ m \ c^2 \frac{\partial^2}{\partial \beta_\mu \partial \beta_\nu} (\frac{K_1(\beta)}{\beta})$$

we get

$$T_o^{\mu\nu} = 4\pi\, m\, \alpha \frac{K_3(\beta)}{\beta} u^\mu u^\nu + 4\pi\, m\, \alpha\, c^2 \frac{K_2(\beta)}{\beta^2} g^{\mu\nu} \quad , \tag{3-59}$$

where we've used the fact that $\frac{\partial \beta^\mu}{\partial \beta_\nu} = g^{\mu\nu}$. Defining μc^2 to be the energy density, and P for pressure and letting

$$\mu + \frac{P}{c^2} = 4\pi\, m\, \alpha \frac{K_3(\beta)}{\beta} \quad , \quad \text{and} \quad \frac{P}{c^2} = 4\pi\, m\, \alpha \frac{K_2(\beta)}{\beta^2} \quad , \tag{3-60}$$

equation 3-59 becomes

$$T_o^{\mu\nu} = (\mu + \frac{P}{c^2}) u^\mu u^\nu + P\, g^{\mu\nu} \quad , \tag{3-61}$$

which is the Energy-Momentum tensor for a perfect fluid. Defining the projection operator $\Delta^{\mu\nu}$ to be

$$\Delta^{\mu\nu} \equiv g^{\mu\nu} + \frac{1}{c^2} u^\mu u^\nu \quad ,$$

(it projects objects down onto the hypersurface orthogonal to the vector u^μ) , equation 3-61 can be rewritten as

$$T_o^{\mu\nu} = \mu\, u^\mu u^\nu + P\, \Delta^{\mu\nu} \; . \tag{3-62}$$

It can also be seen from all this that $u^\mu \equiv \frac{c\beta^\mu}{\beta}$ is the hydrodynamic 4-velocity. Relativistic temperature is defined by

(3-63) $$\beta \equiv \frac{m\,c^2}{K\,T} \quad .$$

From the conservation law (3-34), we get for the perfect fluid

(3-63) $$M_o^{\ \mu} = (\rho u^\mu)_{|\mu} = u^\mu \frac{\partial \rho}{\partial x^\mu} + \rho\, u^\mu{}_{|\mu} = 0 \ .$$

Defining the derivative along a stream line $\frac{d}{d\tau}$ to be

$$\frac{d}{d\tau} = \frac{dx^\mu}{d\tau} \frac{\partial}{\partial x^\mu} = u^\mu \frac{\partial}{\partial x^\mu}$$

we finally arrive at the continuity equation

(3-64) $$\frac{d\rho}{d\tau} + \rho\, u^\mu{}_{|\mu} = 0 \quad .$$

The conservation law $T_o^{\ \lambda\nu}{}_{|\nu} = 0$ becomes

$$T_o^{\ \lambda\nu}{}_{|\nu} = 0 = \Delta_{\lambda\mu}\ T_o^{\ \lambda\nu}{}_{|\nu} \quad .$$

But

$$T_o^{\ \lambda\nu}{}_{|\nu} = (\partial_\nu \mu) u^\lambda u^\nu + \mu\, u^\lambda{}_{|\nu}\, u^\nu + \mu\, u^\lambda u^\nu{}_{|\nu} + (\partial_\nu P)\Delta^{\lambda\nu} + P\, \Delta^{\lambda\nu}{}_{|\nu}$$

$$= (\mu + \frac{P}{c^2}) \frac{\delta u^\lambda}{\delta\tau} + (\partial_\nu P)\ \Delta^{\lambda\nu} \qquad ,$$

Therefore we have

$$\Delta_{\lambda\mu} \; T_o^{\lambda\nu}\Big|_\nu = (\mu + \frac{P}{c^2}) \frac{\delta u_\mu}{\delta\tau} + \Delta^\lambda_{\ \mu} \, \partial_\lambda \, P = 0$$

or

$$(\mu + \frac{P}{c^2}) \frac{\delta u_\mu}{\delta\tau} = - \Delta^\lambda_{\ \mu} \, \partial_\lambda \, P \quad . \tag{3-65}$$

Equation 3-65 is essentially the relativistic analog of Newtons second law, where $(\mu + \frac{P}{c^2})$ is the effective mass.

CHAPTER IV

Non-Equilibrium and Transport Effects

In this chapter we investigate transport effects arising from non-equilibrium. By means of a relativistic version of the Grad Method we obtain explicit representations of the Transport coefficients.

In the last chapter it was shown that for the equilibrium situation we obtained the Energy-Momentum tensor, and the Numerical flux vector of a perfect fluid. In so doing we identified $u^\mu (= c\beta^\mu/\beta)$ with the hydrodynamic 4-velocity. This identification was unique in that u^μ was naturally singled out as a result of the equilibrium condition. As a consequence the Energy-Momentum tensor, and the Numerical Flux vector we expressed entirely in terms of u^μ, and certain scalars. But given a non-equilibrium situation the choice of the Hydrodynamic 4-velocity becomes somewhat arbitrary (1). Because once we've lost equilibrium, we've lost the condition that singled out β^μ. Since we will be interested in viscous effects and the associated transport coefficients it will be necessary to retain conventional Hydrodynamics. This is done by assuming that the deviation from equilibrium of the system is small, so that we can say that at each point in space-time the distribution function can be approximated by a local Maxwellian distribution,

$$N_o(x,P) = \alpha(x) e^{\beta_\mu(x) P^\mu} \qquad (2) \quad .$$

From this it can be seen that $N_o(x,P)$ will vary from point to point, which means that u^μ will also be a function of x .

The problem still remains as to how $N_o(x,P)$ will be fitted to the actual distribution function. This will involve removing a certain amount of the arbitrariness in the five parameters $\alpha(x)$, $\beta^\mu(x)$ occuring in N_o . More on this will be said at a later point in the text. The condition of small deviations from equilibrium is expressed as,

(4-1) $$N(x,P) = N_o(1 + f(x,P)) ,$$

where $f(x,P)$ $(\ll 1)$ is the first order correction to the equilibrium distribution. We will therefore neglect terms second order and up in f . We also impose the condition that

$$\frac{\partial_\nu f}{f} \ll \frac{\partial_\nu N}{N} .$$

Substituting equation 4-1 into the expression for the energy-momentum tensor we get

$$T^{\mu\nu} = mc^2 \int N_o(1+f) P^\mu P^\nu \, d\omega$$

$$= mc^2 \int N_o \, P^\mu P^\nu \, d\omega + mc^2 \int N_o \, f \, P^\mu P^\nu \, d\omega ,$$

which leads to

(4-2) $$T^{\mu\nu} = T_o^{\ \mu\nu} + T_1^{\ \mu\nu} ,$$

where

(4-3) $$T_1^{\mu\nu} = mc^2 \int N_o \, f \, P^\mu P^\nu \, d\omega \quad .$$

Similarly for the Numerical flux vector we get,

(4-4) $$M^\mu = M_o^{\ \mu} + M_1^{\ \mu} \ ,$$

where

(4-5) $$M_1^{\ \mu} = mc \int N_o \, f \, P^\mu \, d\omega \quad .$$

The first order correction terms $M_1^{\ \mu}$, and $T_1^{\ \mu\nu}$ will contain the necessary transport effects. To get a better understanding as to the physics involved, in the preceding statement, let us for the moment approach of non-equilibrium from a more phenomenological point of view. The ideas we will use were first put forth by Eckart (3), and more recently elaborated on by Weinberg (4). In the Eckart approach the arbitrariness involved in the choice of the Hydrodynamic 4-velocity is removed by stipulating that the density ρ is the 4th component of M^μ in the comoving frame, in other words

(4-4) $$u_\mu M^\mu = u_\mu M_o^{\ \mu} \ ,$$

also that the energy density μ is the (4,4) component of $T^{\mu\nu}$ as measured in the comoving frame, in other words

(4-5) $$u_\mu u_\nu T^{\mu\nu} = u_\mu u_\nu T_o^{\mu\nu} ,$$

and by finally choosing the space-time direction of u^μ to be that of the particle current, i.e.,

(4-6) $$u^\mu = \frac{c}{(-M^\alpha M_\alpha)^{1/2}} M^\mu .$$

By means of $\Delta^{\mu\nu}$, and u^μ we now proceed to decompose the energy-momentum tensor into its proper components, i.e.

(4-7) $$\mu \equiv u_\alpha u_\beta T^{\alpha\beta}$$

(4-8) $$Q^\mu \equiv -\Delta^\mu{}_\alpha T^{\alpha\beta} u_\beta$$

(4-9) $$\epsilon^{\mu\nu} \equiv \Delta^\mu{}_\alpha \Delta^\nu{}_\beta T^{\alpha\beta} .$$

From equations 4-7,8,9 we get the following identity

(4-10) $$T^{\mu\nu} = \mu u^\mu u^\nu + Q^\mu u^\nu + Q^\nu u^\mu + \epsilon^{\mu\nu} .$$

The components of $T^{\mu\nu}$ (equations 4-7,8,9) have an important physical significance, and through them equation 4-10 reveals the structure of $T^{\mu\nu}$. To make this physical significance more transparent let us confine our analysis to the local rest frame i.e. choose the time axis to be along u^μ. Then using equation 4-5 we have that μ as defined by equation 4-7 becomes

(4-11) $$\mu = T_o^{44}$$

which is just the invariant energy density.

Defining what is commonly designated as the viscous stress tensor (5), $\tau^{\mu\nu}$, through the relation

(4-12) $$\epsilon^{\mu\nu} = \tau^{\mu\nu} + P\Delta^{\mu\nu} ,$$

where P is the local equilibrium pressure, equation 4-10 becomes

(4-13) $$T^{\mu\nu} = T_o^{\mu\nu} + Q^\mu u^\nu + Q^\nu u^\mu + \tau^{\mu\nu}$$

where we have used equation 3-61. Comparing equation 4-13 to equation 4-2 we now see that the phenomenological addition $T_1^{\mu\nu}$ to the perfect fluid energy momentum tensor $T_o^{\mu\nu}$ is

(4-14) $$T_1^{\mu\nu} = Q^\mu u^\nu + Q^\nu u^\mu + \tau^{\mu\nu} ,$$

which includes heat conduction, and dissipative effects due to viscosity. To get more explicit representations of Q^μ, and $\tau^{\mu\nu}$ let us once again confine our analysis to the local rest frame. The usual phenomenological arguments given in ascertaining the general form of τ^{ij} (i,j,=1,2,3) are essentially (5,6) :

(a) τ^{ij} must depend on the space derivatives of velocity, since its the relative motion of the fluid parts that cause internal friction , i.e. viscosity.

(b) The velocity gradients are assumed small, so that we can expect that τ^{ij} depends on the first derivatives.

(c) τ^{ij} cannot be independent of $\frac{\partial u^i}{\partial x^k}$ since $\tau^{ij} = 0$ when $u^i = \text{const.}$

(d) As a consequence of b , the derivatives should appear linearly.

(e) If the fluid is in uniform rotation then there will be no relative motion hence no viscosity. We know that for uniform rotation that the rate-of-strain tensor $\frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i} = 0$. Thus sums of the form $\frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i}$ should appear. Note that this expression satisfies conditions a,b,c,d .

Hence the most general form of τ^{ij} satisfying conditions a,-d is

$$\tau^{ij} = -\nu\left(\frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i} - \frac{2}{3}\, \underset{\sim}{\nabla}\cdot\underset{\sim}{u}\, \delta^{ij}\right) - \kappa\underset{\sim}{\nabla}\cdot\underset{\sim}{u}\delta^{ij} \tag{4-15}$$

ν is the shear viscosity, and κ is the bulk viscosity. The extra term of $-\frac{2}{3}\underset{\sim}{\nabla}\cdot\underset{\sim}{u}\,\delta^{ij}$ in equation 4-15 is put in just to make the quantity in paranthesis traceless. In general

$$\tau^{\mu\nu} = -\nu\,\Delta^{\mu}{}_{\alpha}\Delta^{\nu}{}_{\beta}\left(u^{\alpha|\beta} + u^{\beta|\alpha} - \frac{2}{3}\Delta^{\alpha\beta}u^{\lambda}{}_{|\lambda}\right) - \kappa\,\Delta^{\mu\nu}u^{\lambda}{}_{|\lambda} \,. \tag{4-16}$$

For future reference we make the following definition

$$\sigma^{\mu\nu} \equiv \Delta^{\mu}{}_{\alpha}\Delta^{\nu}{}_{\beta}(u^{\alpha|\beta} + u^{\beta|\alpha} - \frac{2}{3}\Delta^{\alpha\rho}u^{\lambda}{}_{|\lambda}) \quad . \tag{4-17}$$

Equation 4-16 now becomes

$$\tau^{\mu\nu} = -\nu\,\sigma^{\mu\nu} - \kappa\,\Delta^{\mu\nu}\,u^{\lambda}{}_{|\lambda} \quad . \tag{4-18}$$

To obtain the phenomenological form of the heat flux vector we once again refer to the arguments given by Weinberg ( ) . To obtain the structure of q^{μ} we once again for simplicity consider the problem in the locally co-moving Lorentz frame i.e. $u^{\mu} = (0,0,0,c)$. Thus from equation (4-14) we can see that Q^{i} $(i = 1,2,3)$ is just $T_{1}{}^{i4}$. The structure we pick for Q^{i} must be such that it reduces to the classical Fourier conduction Law under suitable conditions. The most general form which will yield positive entropy generation ( ), i.e. $S^{\alpha}{}_{|\alpha} \geq 0$, is

$$Q^{i} = T_{1}{}^{i4} = -\chi\left(\frac{\partial T}{\partial x^{i}} + T\,\frac{\partial u^{i}}{\partial t}\right) \quad .$$

Thus in general we can write

$$Q^{\mu} = -\chi\,\Delta^{\mu\alpha}\left[\frac{\partial T}{\partial x^{\alpha}} + T\,\dot{u}_{\alpha}\right] \quad . \tag{4-19}$$

Notice that for small accelerations 4-14 reduces to $\underset{\sim}{Q} = -\chi\nabla T$. The

relativistic version of the heat flow vector gives rise to a result not occuring in the classical case, i.e. the term $T\,\dot{u}_\alpha$ gives rise to isothermal heat flow in accelerated matter, in the opposite direction to the acceleration.

For our future calculations we will use an alternative ( ) definition of Q^μ given by Israel ( ) :

$$(4\text{-}20) \qquad Q^\mu = \lambda\,\Delta^{\mu\nu}\,\partial_\nu\theta$$

where

$$\theta \equiv \frac{1}{c^2 T}\left(\frac{\mu c^2}{\rho} + \frac{P}{\rho} - TS\right) .$$

In the preceding discussions on the phenomenological structure of Q^μ and $\tau^{\mu\nu}$ we followed the approach of Eckart. In so doing the arbitrariness in the hydrodynamic 4-velocity was removed by imposing equation 4-4,5,6. In what will follow we will not impose all of these conditions. We will still retain equations 4-4,5 but not equation 4-6, i.e. we will not assume that u^μ and M^μ have the same space-time direction. As a consequence of this the definition of Q^μ will have to be modified. We will define heat flow to be the flow of energy relative to the particle stream (8), i.e.

$$(4\text{-}21) \qquad Q^\mu \equiv -\,c\,\frac{1}{(-M_\alpha M^\alpha)^{1/2}}\left[T^{\mu\nu}M_\nu - \frac{1}{(-M_\beta M^\beta)}\,M^\mu T^{\lambda\nu}\,M_\lambda M_\nu\right] .$$

Comparing equation 4-21 with equation 4-8 we see that equation 4-21 would reduce to equation 4-8 if we were to retain equation 4-6. Since we will be concerned with quantities expressed only up to first order in f, Q^μ expressed to first order in f is,

$$Q^\mu = - \eta \, c^2 M_1{}^\mu - \Delta^\mu{}_\rho \, T_1{}^{\rho\nu} u_\nu \qquad (9) \tag{4-22}$$

where

$$\eta \equiv \frac{\mu + \frac{P}{c^2}}{\rho} \quad .$$

The viscous stress tensor will be given by

$$\tau^{\mu\nu} = \Delta^\mu{}_\alpha \Delta^\nu{}_\beta \, T_1{}^{\alpha\beta} \quad . \tag{4-23}$$

Equation 4-23 follows directly from equations 4-9,12.

The viscous and heat flow effects are contained in $M_1{}^\mu$ and $T_1{}^{\mu\nu}$. But to get expressions for these we still have to solve for the function f . To solve for f we will consider the linearized Boltzmann equation. To obtain the linearized Boltzman equation one substitutes into the Boltzmann equation (i.e. equation 2-32) the first order stipulations associated with equation 4-1. Upon performing this substitution we have,

$$(4\text{-}24) \qquad P^{\mu}\partial_{\mu}\,(N_o + N_o f) = \iiint W\,\{(N_o{}^{*} + N_o{}^{*}f^{*})('N_o{}^{*} + 'N_o{}^{*}\,'f^{*})$$

$$- (N_o + N_o f)('N_o + 'N_o\,'f)\}\; d'\omega\; d\omega^{*}\; d'\omega^{*} \quad .$$

Expanding out equation 4-24 we have,

$$(4\text{-}25) \qquad P^{\mu}\partial_{\mu}N_o + P^{\mu}N_o\partial_{\mu}f + P^{\mu}f\partial_{\mu}N_o = \iiint W\,\{N_o\,'N_o{}^{*} + {}^{*}N_o{}'N_o{}^{*}(f^{*}+'f^{*}) +$$

$$+ f^{2}(N_o{}^{*}\,'N_o{}^{*}) - N_o{}'N_o - N_o{}'N_o(f + 'f) - f^{2}(N_o{}'N_o)\} \qquad \times$$

$$\times\; d'\omega\; d\omega^{*}\; d'\omega^{*} \quad .$$

Since we are only retaining terms up to first order in f equation 4-25 reduces to

$$(4\text{-}26) \qquad P^{\mu}\partial_{\mu}N_o = \iiint W\,\{(N_o{}^{*}\,'N_o{}^{*} - N_o{}'N_o) + N_o{}^{*}\,'N_o{}^{*}(f^{*} + 'f^{*}) \; -$$

$$- N_o{}'N_o(f+'f)\}d'\omega\; d\omega^{*}\; d'\omega^{*} \; .$$

But we know that N_o is the local equilibrium distribution function hence it must satisfy equation 3-45, so that,

$$N_o{}^{*}\;'N_o{}^{*} = N_o\;'N_o \quad .$$

In view of this equation 4-26 becomes

$$(4\text{-}27) \qquad P^{\mu}\partial_{\mu}\,N_o = -\,N_o\iiint W\;'N_o\;(f+'f-f^{*}-'f^{*})\; d'\omega\; d\omega^{*}\; d'\omega^{*} \quad .$$

Defining $\delta(f) \equiv f + 'f - f^* - 'f^*$, and defining the linear integral operator L to be

$$L(f) \equiv \iiint W(P,'P:P^*,'P^*)\ 'N_o\ \delta(f)\ d'\omega\ d\omega^*\ d'\omega^* \quad ,$$

equation 4-27 simple becomes

$$P^\mu\ \partial_\mu\ N_o(x,P) = -N_o(x,P)\ L(f) \quad . \tag{4.28}$$

It can easily be seen that,

$$\int N_o(x,P)G(x,P)L(F)d\omega = \frac{1}{4} \iiiint W\ N_o 'N_o\ \delta(F)\delta(G)\ d\omega\ d'\omega\ d\omega^*\ d'\omega^* \ . \tag{4-29}$$

Thus multiplying equation 4-28 by "1" and integrating over $d\omega$ we can see that inview of equation 4-29 we get,

$$M_o{}^{\mu}{}_{|\mu} = 0 \quad , \tag{4-30}$$

where we have used the fact that $\delta(1) = 0$. By multiplying equation 4-28 by P^ν and integrating over $d\omega$ we get

$$T_o{}^{\mu\nu}{}_{|\nu} = 0 \quad , \tag{4-31}$$

where we've used equation 4-29, and the fact that $\delta[P^\nu] = 0$.

By equation 4-28 it is seen that in order to solve for $f(x,P)$ we have to specify the space-time gradients of α, β_μ i.e. $\partial_\mu N_o$. But in view of equations 4-30, 31 we see that all these gradients cannot be freely assigned, i.e. they are subject to the five conditions expressed in equations 4-30, 31. The ramifications of equations 4-30, 31 have already been derived in chapter III, (see equations 3-64, 65).

To obtain a solution for f we make use of the Relativistic version of the Grad approximation method (1) as was suggested by Anderson (11). What one essentially does is to construct an orthogonal set of polynomials in momentum by means of the Gram-Schmidt Orthogonalization process, where we use the local equilibrium distribution function, N_o, as the weighting function. For example designating the polynomials by $\overset{m}{H}{}^{\ell_1,\ldots,\ell_m}$, where m is the order of the polynomial and $(\ell_1,\ldots,\ell_m)$ is the appropriate tensor index, then we have

$$\overset{o}{H} = 1 \ , \quad \overset{1}{H}{}^\mu = P^\mu - a^\mu \overset{o}{H} \ , \quad \overset{2}{H}{}^{\mu\nu} = P^\mu P^\nu - a^{\mu\nu}{}_\rho \overset{1}{H}{}^\rho - a^{\mu\nu} \overset{o}{H} \ ,$$

etc., and where a^μ, $a^{\mu\nu}{}_\rho$, $a^{\mu\nu}$, etc. are determined by the orthogonality conditions (12). Once the orthogonal set is constructed we expand N as

$$N = N_o \left(\sum_{m=0}^{\infty} \overset{m}{a}_{\ell_1,\ldots,\ell_m}(x) \, \overset{m}{H}{}^{\ell_1,\ldots,\ell_m}(x,P) \right) \quad .$$

The significant and crucial step of the Grad approximation method is to set all coefficients, $\overset{m}{a}_{\ell_1,\ldots,\ell_m}$, past a certain order equal to zero. For example suppose we set all the coefficients $\overset{m}{a}_{\ell_1,\ldots,\ell_m}$ equal to zero for $m \geq 2$, then equation 4-40 can be essentially written as,

$$N = N_o(A(x) + B_\mu(x)\, P^\mu) \quad . \tag{4-33}$$

But his just reproduces the perfect fluid theory. Thus we are led to the approximation of setting all coefficients, $\overset{m}{a}_{\ell_1,\ldots,\ell_m}$, equal to zero for $m \geq 3$. The obvious question to ask at this point is; "Why this order of approximation and not higher?". Andersons answers this by saying that, "the best that one can do at present is to increase the order of the approximation and calculate the effect on whatever quantities one is interested in computing. This at least in the Grad Method is doable <u>although the effort involved is rather enormous</u>. Alternately, one can try to compare predictions of the approximation with experiment. At least in the classical case these predictions agree remarkably well with experiment". (13) Using this approximation and comparing equation 4-1 with equation 4-32 we can write without loss of generality that

$$f = C_5(x) + C_\mu(x)\, P^\mu + b_{\mu\nu}\, P^\mu P^\nu \quad . \tag{4-34}$$

To obtain the third moment equation which will relate the transport effects we first multiply equation 4-27 by $P^\lambda P^\mu$ and integrate over the momentum i.e.

$$(4\text{-}35) \qquad P^\lambda P^\mu P^\nu \partial_\nu N_o = - \iiiint W \, 'N_o \, N_o \, \delta(f) \, P^\lambda P^\mu \, d\omega \, d'\omega \, d\omega^* \, d'\omega^* \quad .$$

Using the symmetries of W and the fact that $N_o \, 'N_o = N_o^* \, 'N_o^*$ equation 4-35 can be written as,

$$(4\text{-}36) \qquad \partial_\mu \int N_o \, P^\lambda P^\mu P^\nu \, d\omega = - \frac{1}{4} \iiiint W \, 'N_o \, N_o \, \delta(f) \, \delta(P^\lambda P^\mu) d\omega \, d'\omega \, d\omega^* \, d'\omega^* \, .$$

Substituting equation 4-34 into equation 4-36 we get,

$$\begin{aligned}(4.37) \qquad \partial_\nu \, \mathcal{U}_o^{\lambda\mu\nu} = & - \frac{1}{4} \, m \, c^3 \iiint W \, N_o \, 'N_o \, \delta(C_5(x)) \, \delta(P^\lambda P^\mu) \, d\omega \, d'\omega \, d\omega^* \, d'\omega^* \\ & - \frac{1}{4} \, m \, c^3 \iiiint W \, N_o \, 'N_o \, C_\nu(x) \, \delta(P^\nu) \, \delta(P^\lambda P^\mu) \, d\omega \, d'\omega \, d\omega^* \, d'\omega^* \\ & - \frac{1}{4} \, b_{\alpha\beta}(x) \, mc^3 \iiiint W \, N_o \, 'N_o \, \delta(P^\alpha P^\beta) \, \delta(P^\lambda P^\mu) \, d\omega \, d'\omega \, d\omega^* \, d'\omega^*\end{aligned}$$

where we've made use of the following definition:

$$(4\text{-}38) \qquad \mathcal{U}_o^{\lambda\mu\nu} \equiv m \, c^3 \int P^\lambda P^\mu P^\nu \, N_o \, d\omega \quad .$$

Since $C_5(x)$ and P^ν are collisional invariants then equation 4-37 simplifies to

(4-39) $$\partial_\nu \mathcal{U}_o^{\lambda\mu\nu} = - b_{\alpha\beta} X^{\alpha\beta\lambda\mu}$$

where

$$X^{\alpha\beta\lambda\mu} \equiv \frac{1}{4} m c^3 \iiiint W N_o 'N_o \delta(P^\alpha P^\beta) \delta(P^\lambda P^\mu) d\omega d'\omega d\omega^* d'\omega^* \qquad (14) .$$

Equation (4-39) contains all the necessary transport effects. From this we will proceed to derive expressions for the transport coefficients. But before proceeding into the explicit calculations, it is best that the essential overall plan be described so that the reader can appreciate the rather beautiful simplicity of the results when they do finally appear. We will express the left hand side of equation 4-39 in terms of gradients of thermodynamic parameters, and of Hydrodynamic velocity. In particular we will have terms like, $\sigma_{\alpha\beta}$ (see equation 4-17), $\Delta^\mu{}_\alpha \theta_{,\mu}$, etc. Through equations 4-22, 23 we will be able to express $b_{\alpha\beta}$ in terms of $\tau_{\alpha\beta}$, and Q_α . Thus we transform equation 4-39 into linear combinations of $\sigma_{\alpha\beta}, \Delta^{\alpha\beta}\theta_{,\beta}$, etc. on the left hand side and $\tau_{\alpha\beta}$, and Q_α on the right hand side. By properly identifying corresponding terms on each side we end up with

$$\tau_{\alpha\beta} = ? \ \sigma_{\alpha\beta} + ? \ \Delta_{\alpha\beta} u^\mu{}_{|\mu}$$

$$Q^\mu = ? \ \Delta^{\mu\rho} \partial_\rho \theta \qquad .$$

But these are just the phenomenological expressions for the viscous stress tensor, and heat flux vector we've derived (equations 4-18,

4-20). Thus we can identify the coefficients (represented by ?) to be the respective transport coefficients. Of course the calculations are far more complicated, but bearing the above in mind the calculations that follow will seem clearer.

We will first put the left hand side of equation 4-39 into a more convenient form. Starting with equation 4-38, we see that in terms of our generating function we have

$$\mathcal{U}_o^{\lambda\mu\nu} = m\, c^3 \frac{\partial^3 Z_o}{\partial\beta_\lambda \partial\beta_\mu \partial\beta_\nu} \quad . \tag{4-40}$$

From equation 4-40 we get

$$\mathcal{U}_o^{\lambda\mu\nu} = \xi\, u^\lambda u^\mu u^\nu + \zeta(\Delta^{\lambda\mu} u^\nu + \Delta^{\mu\nu}\, u^\lambda + \Delta^{\nu\lambda}\, u^\mu) \tag{4-41}$$

where

$$\begin{aligned} \xi + \frac{3\zeta}{c^2} &= 4\pi\, m\alpha \frac{K_4(\beta)}{\beta} \\ \frac{\zeta}{c^2} &= 4\pi\, m\alpha \frac{K_3(\beta)}{\beta^2} \quad . \end{aligned} \tag{4-42}$$

Thus the expression for $u_o^{\lambda\mu\nu}|_\nu$ can be written as

$$\begin{aligned} \mathcal{U}_o^{\lambda\mu\nu}|_\nu = {} & (\dot{\xi} + \xi\, u^\alpha|_\alpha + \frac{2}{c^2}\dot{\zeta} + \frac{2}{c^2}\zeta\, u^\alpha|_\alpha) u^\lambda u^\mu \\ & + (\xi\, \dot{u}^\mu + \zeta^{|\mu} + \frac{3\zeta}{c^2}\dot{u})u^\lambda + (\xi\, \dot{u}^\lambda + \zeta^{|\lambda} + \frac{3\zeta}{c^2}\dot{u}^\lambda)\, u^\mu \end{aligned} \tag{4-43}$$

$$+ (\dot{\zeta} + \zeta\, u^{\alpha}{}_{|\alpha})\Delta^{\lambda\mu} + \zeta(u^{\lambda|\mu} + u^{\mu|\lambda}) \quad .$$

Using the facts that,

$$\Delta^{\lambda}{}_{\rho}\, \Delta^{\mu}{}_{\alpha}\, (u^{\rho|\sigma} + u^{\sigma|\rho}) = u^{\lambda|\mu} + u^{\mu|\lambda} + \frac{\dot{u}^{\mu} u^{\nu}}{c^2} + \frac{\dot{u}^{\lambda} u^{\mu}}{c^2} \quad ,$$

and that

$$\Delta^{\lambda\mu} = \Delta^{\lambda}{}_{\rho}\, \Delta^{\mu}{}_{\sigma}\, \Delta^{\rho\sigma}$$

equation 4-43 becomes

$$(4\text{-}44) \qquad U_o{}^{\lambda\mu\nu}{}_{|\nu} = \zeta\sigma^{\lambda\mu} + [(\xi + \frac{2\zeta}{c^2})\dot{u}^{\mu} + \Delta^{\mu}{}_{\alpha}\, \zeta^{|\alpha}]u^{\lambda}$$

$$+ [(\xi + \frac{2}{c^2})\dot{u}^{\lambda} + \Delta^{\lambda}{}_{\alpha}\, \zeta^{|\alpha}]u^{\mu} + [\dot{\xi} + (\xi + \frac{2\zeta}{c^2})u^{\alpha}{}_{|\alpha}]u^{\lambda}u^{\mu} +$$

$$+ (\dot{\zeta} + \frac{5}{3}\,\zeta\, u^{\alpha}{}_{|\alpha})\Delta^{\lambda\mu} \quad ,$$

where we've made use of equation 4-17, and the fact that

$$\zeta^{|\mu} = \Delta^{\mu}{}_{\alpha}\, \zeta^{|\alpha} - \frac{\dot{\zeta}\, u^{\mu}}{c^2} \quad .$$

Let us for the moment consider the coefficient of the u^{λ} term. Making use of equation 3-65 the coefficient becomes

$$(4\text{-}45) \qquad q^{\mu} \equiv - \frac{(\xi + \frac{2\zeta}{c^2})}{(\mu + \frac{P}{c^2})}\, \Delta^{\mu}{}_{\alpha}\, P^{|\alpha} + \Delta^{\mu}{}_{\alpha}\, \zeta^{|\alpha} = \Delta^{\mu}{}_{\alpha}\, [- \frac{(\xi + \frac{2\zeta}{c^2})}{(\mu + \frac{P}{c^2})}\, P^{|\alpha} + \zeta^{|\alpha}] \quad .$$

From (4-42) we see that

$$(\xi + \frac{2\zeta}{c^2}) = 4\pi\ m\alpha\ (\frac{K_4}{\beta} - \frac{K_3}{\beta^2}) \quad . \tag{4-46}$$

Using equation 3-60, and equation 4-46 we see that

$$\frac{(\xi + \frac{2\zeta}{c^2})}{(\mu + \frac{P}{c^2})} = \frac{K_4(\beta)}{K_3(\beta)} - \frac{1}{\beta} \quad . \tag{4-47}$$

From the fact that $P = 4\pi\ m\ c^2\alpha\ \frac{K_2(\beta)}{\beta^2}$ we get

$$P|^{\lambda} = -\ 4\pi\ m\ c^2\alpha\ \frac{K_3}{\beta^2}\ \beta|^{\lambda} + 4\pi\ m\ c^2\ \frac{K_2}{\beta^2}\ \alpha|^{\lambda} \quad , \tag{4-48}$$

and equation 4-42 we get,

$$\zeta|^{\lambda} = 4\pi\ m\ c^2\alpha\ \ (\frac{1}{\beta} - \frac{K_4}{3})\ \frac{K_3}{\beta^2}\ \beta|^{\lambda} + 4\pi\ m\ c^2\ \frac{K_3}{\beta^2}\ \alpha|^{\lambda} \quad . \tag{4-49}$$

Substituting equations 4-47, 48, 49 into equation 4-45 we obtain

$$q^{\mu} = \Delta^{\mu}{}_{\gamma}\ 4\pi\ m\ c^2\alpha\,\{\frac{K_3}{\beta^2} - \frac{K_2}{\beta^2}\ (\frac{K_4}{K_3} - \frac{1}{\beta})\}\ \frac{\alpha|^{\gamma}}{\alpha} \quad . \tag{4-50}$$

But we know that

$$\mathrm{Log}\ \alpha = \frac{m\ c^2}{K}\ \theta + \mathrm{const.} \quad (15) \tag{4-51}$$

Thus equation 4-50 becomes

$$q^{\mu} = \frac{m\,c^2}{K}\left[\zeta - P\,\frac{\left(\xi + \frac{2\zeta}{c^2}\right)}{\left(\mu + \frac{P}{c^2}\right)}\right]\Delta^{\mu}{}_{\alpha}\,\theta^{|\alpha} \quad (4\text{-}52)$$

Defining the relativistic enthalpy η to be,

$$\eta \equiv \frac{\mu + \frac{P}{c^2}}{\rho} = \frac{K_3}{K_2} ,$$

we are able to obtain the rather obvious identities

$$\zeta = \eta\,P \quad , \quad \xi + \frac{2\zeta}{c^2} = \rho + \frac{5\eta\,P}{c^2} \quad . \quad (4\text{-}53)$$

Using these identities, and the fact that $\frac{P}{c^2} = \frac{\rho}{\beta}$ equation 4-52 becomes

$$q^{\mu} = \frac{P}{\eta}\left[\eta^2 - \frac{5}{\beta}\,\eta - 1\right]\Delta^{\mu}{}_{\alpha}\,\theta^{|\alpha} \quad , \quad (4\text{-}54)$$

where we've used the following results given by Israel (16);

$$\frac{d\eta}{d\beta} = \eta^2 - \frac{5}{\beta}\,\eta - 1 \quad , \quad \text{and}$$

(4-55)

$$\frac{\gamma}{\gamma-1} = -\,\beta^2\,\frac{d\eta}{d\beta} \quad ,$$

where γ is the ratio of specific heats, i.e.

$$\gamma = \frac{C_P}{C_V} \quad .$$

Let us next consider the coefficient of $u^\lambda u^\mu$ in equation 4-44, i.e.

$$\dot{\xi} + (\xi + \frac{2\zeta}{c^2})\, u^\alpha{}_{|\alpha} \quad .$$

It can easily be seen that

$$\xi = \rho + \frac{3\zeta}{c^2} \quad ,$$

from which we can write

(4-56) $$\dot{\xi} = \dot{\rho} + \frac{3\dot{\zeta}}{c^2} \quad .$$

From equation 4-53 we can write

(4-57) $$\dot{\zeta} = \dot{\eta}\, P + \eta\, \dot{P} \quad .$$

Along the streamlines entropy is conserved (17) i.e. $\dot{s} = 0$. Thus we can write

$$\dot{\eta} = (\frac{\partial \eta}{\partial P})_s\, \dot{P} \quad ,$$

from which it follows that

(4-58) $$\dot{\eta} = -\frac{\gamma}{\beta}\, u^\lambda{}_{|\lambda} \quad (18) \quad .$$

One can also show that

$$\dot{P} = \rho\, c^2\, T\dot{\theta} - \eta\, c^2\, \rho\, \frac{\dot{\beta}}{\beta} \tag{4-59}$$

$$= \{P[\eta\beta(\gamma-1) - \gamma] - \eta\, P\, \beta(\gamma-1)\}\, u^{\alpha}{}_{|\alpha} \quad . \qquad (19)$$

Using equations 4-57, 58, 59 equation 4-56 becomes

$$\dot{\xi} = \dot{\rho} + \frac{3}{c^2}\{-\eta\beta\gamma - \frac{P\gamma}{\beta}\}\, u^{\alpha}{}_{|\alpha} \quad . \tag{4-60}$$

Using equation 4-60, the fact that $(\xi + \frac{2\zeta}{c^2}) = (\rho + \frac{5\eta\, P}{c^2})$ and also equation 3-64, the coefficient of $u^{\lambda}u^{\mu}$ becomes,

$$-\frac{P\eta}{c^2}[3\gamma - 5 + \frac{3\gamma}{\eta\beta}]\, u^{\alpha}{}_{|\alpha} \equiv A_2\, u^{\alpha}{}_{|\alpha} \quad .$$

By a similar calculation the coefficient of the $\Delta^{\lambda\mu}$ term in equation 4-44 becomes

$$-\frac{1}{3}\,\eta\, P[3\gamma - 5 + \frac{3\gamma}{\eta\beta}]\, u^{\alpha}{}_{|\alpha} \equiv A_3\, u^{\alpha}{}_{|\alpha} \quad .$$

Thus equation 4-44 can finally be written as

$$U_o{}^{\lambda\mu\nu}{}_{|\nu} = \eta\, P\, \sigma^{\lambda\mu} + q^{\mu}u^{\lambda} + q^{\lambda}u^{\mu} + A_2 u^{\alpha}{}_{|\alpha}\, u^{\lambda}u^{\mu} + A_3\, u^{\alpha}{}_{|\alpha}\, \Delta^{\lambda\mu} \tag{4-61}$$

where

where

$$q^{\mu} \equiv - \frac{P}{\eta} \frac{1}{\beta^2} \left(\frac{\gamma}{\gamma-1}\right) \Delta^{\mu}{}_{\alpha} \, \theta^{|\alpha}$$

$$A_2 \equiv - \frac{P\eta}{c^2} \left[3\gamma - 5 + \frac{3\gamma}{\eta\beta}\right]$$

$$A_3 \equiv - \frac{1}{3} \eta \, P \left[3\gamma - 5 + \frac{3\gamma}{\eta\beta}\right] \quad .$$

Next let us consider the right hand side of equation 4-39 in particular let us consider the structure of $X^{\lambda\mu\alpha\beta}$. Our task now is to ascertain the general form of $X^{\lambda\mu\alpha\beta}$. From the definition of $X^{\alpha\beta\lambda\mu}$ (see equation 4-47) one can see that there is no special spatial direction singled out. Thus we ask ourselves what is the most general form that we can choose and of what quantities will it be composed of. Clearly the only basic quantities at our disposal are u^{μ} , and $\Delta^{\mu\nu}$. As was stated earlier the quantity $\Delta^{\mu\nu}$ projects other tensors onto the 3-flat orthogonal to u^{μ} . Thus the most general combination of these quantities which give us a 4th order tensor, and also satisfies the symmetry properties of $X^{\alpha\beta\lambda\mu}$ is

$$\begin{aligned}(4\text{-}63) \quad X^{\lambda\mu\alpha\beta} &= A\, u^{\lambda}u^{\mu}u^{\alpha}u^{\beta} + B(\Delta^{\lambda\alpha}\Delta^{\mu\beta} + \Delta^{\mu\alpha}\Delta^{\lambda\mu}) \\ &+ C(u^{\lambda}\Delta^{\mu\alpha}u^{\beta} + u^{\mu}\Delta^{\lambda\alpha}u^{\beta} + u^{\lambda}\Delta^{\mu\beta}u^{\alpha} + u^{\mu}\Delta^{\lambda\mu}u^{\alpha}) \\ &+ D\,\Delta^{\lambda\mu}\Delta^{\alpha\beta} + E(\Delta^{\lambda\mu}u^{\alpha}u^{\beta} + \Delta^{\alpha\beta}u^{\lambda}u^{\mu}) \quad .\end{aligned}$$

It should be noted that there are only 3 independent components i.e. D , and E can be expressed in terms of A , B , and C .

It can be seen that

$$A = c^{-8} \, u_\lambda u_\mu u_\alpha u_\beta \, X^{\alpha\beta\lambda\mu} \tag{4-64}$$

$$C = \frac{1}{3} c^{-4} \, u_\lambda u_\beta \Delta_{\mu\alpha} \, X^{\lambda\mu\alpha\beta} \; . \tag{4-65}$$

To find what B is we first multiply equation 4-61 by $\Delta_{\alpha\beta}\Delta_{\lambda\mu}$, i.e.

$$\Delta_{\alpha\beta}\Delta_{\lambda\mu} \, X^{\alpha\beta\lambda\mu} = B(\Delta^\lambda{}_\beta \Delta^\beta{}_\lambda + \Delta^\mu{}_\beta \Delta^\beta{}_\mu) + D \, \Delta^{\alpha\beta}\Delta^{\lambda\mu}\Delta_{\alpha\beta}\Delta_{\lambda\mu} \tag{4-66}$$

Equation 4-65 simplifies to

$$\Delta_{\alpha\beta}\Delta_{\lambda\mu} \, X^{\alpha\beta\lambda\mu} = 6B + 9D \quad , \tag{4-67}$$

since $\Delta^\lambda{}_\alpha \Delta^\alpha{}_\lambda = 3$.

Similarly we can show that,

$$\Delta_{\lambda\alpha}\Delta_{\mu\beta} \, X^{\alpha\beta\lambda\mu} = 12B + 3D \quad . \tag{4-68}$$

Thus from equations 4-67, 68 we see that,

(4-69) $$B = \frac{1}{10} (\Delta_{\lambda\alpha}\Delta_{\mu\beta} - \frac{1}{3} \Delta_{\alpha\beta}\Delta_{\lambda\mu}) X^{\alpha\beta\lambda\mu} .$$

Multiplying equation 4-62 by $u_\alpha u_\beta \Delta_{\lambda\mu}$ we get,

(4-70) $$u_\alpha u_\beta \Delta_{\lambda\mu} X^{\lambda\mu\alpha\beta} = 3 c^4 E .$$

But from the definition of $X^{\alpha\beta\lambda\mu}$ it follows that $g_{\lambda\mu} X^{\lambda\mu\alpha\beta} = 0$, since $\delta(P_\mu P^\mu) = 0$. So that equation 4-70 becomes

(4-71) $$u_\alpha u_\beta \frac{u_\lambda u_\mu}{c^2} X^{\alpha\beta\lambda\mu} = 3 c^4 E .$$

But in view of equation (4-64) we finally have

(4-72) $$E = \frac{1}{3} c^2 A .$$

Similarly from equations 4-66, 63 it follows that,

(4-73) $$D = \frac{1}{9} c^4 A - \frac{2}{3} B .$$

Thus equations 4-71, 72 show that there are only 3 independent components in equation 4-63.

Our next task is essentially to get $b_{\alpha\beta}$ in terms of $\tau_{\alpha\beta}$, and Q^α . As was mentioned earlier this will be accomplished through equations 4-22, 23. Before doing this let us first express $M_1^{\ \mu}$, and $T_1^{\ \mu\nu}$ in terms of $b_{\alpha\beta}$ and moments of N_o . Substituting equation 4-34

into equation 4-5 we have

$$M_1^{\ \mu} = m\ c \int N_o\ f\ P^\mu\ d\omega = m\ c \int N_o (C_5 + C_\nu P^\nu + b_{\alpha\beta}\ P^\alpha P^\beta)\ P^\mu\ d\omega \quad ,$$

which simplifies to

$$(4\text{-}74) \qquad M_1^{\ \mu} = \frac{1}{c^2}\, b_{\alpha\beta}\ \mathcal{U}_o^{\ \alpha\beta\mu} + \frac{1}{c}\, C_\nu\ T_o^{\ \mu\nu} + C_5\ M_o^{\ \mu} \quad .$$

Similarly, for $T_1^{\ \mu\nu}$ we get

$$(4\text{-}75) \qquad T_1^{\ \mu\nu} = \frac{1}{c^2}\, b_{\alpha\beta}\ V_o^{\ \alpha\beta\mu\nu} + \frac{1}{c}\, C_\alpha\ \mathcal{U}_o^{\ \alpha\mu\nu} + C_5\ T_o^{\ \mu\nu}$$

where the fourth moment of N_o , $V_o^{\ \alpha\beta\mu\nu}$, is defined as

$$(4\text{-}76) \qquad V_o^{\ \alpha\beta\mu\nu} \equiv m\ c^4 \int N_o\ P^\alpha P^\beta P^\mu P^\nu\ d\omega \quad .$$

But in equations 4-74, 75 we see that we still have the unknowns C_ν , and C_5 to account for. Through our fitting conditions (i.e. equations 4-4, 5) we will be able to express C_5 and $C_\mu\ u^\mu$ in terms of $b_{\alpha\beta}$. It should be noted that all we can fix is $C_\mu\ u^\mu$. The spatial part of C_μ remains arbitrary; which is a result of the degree of arbitrariness we've left in u^μ , (i.e. in not picking a particular space-time direction for u^μ). From one of our fitting conditions, equation 4-4, we see that

$$(4\text{-}77)\qquad 0 = M_1{}^{\mu} u_\mu = \frac{1}{c^2}\, b_{\alpha\beta}\, U_o{}^{\alpha\beta\mu} u_\mu + \frac{1}{c}\, C_\nu\, u_\mu\, T_o{}^{\mu\nu} + C_5\, u_\mu\, M_o{}^{\mu} \quad .$$

Substituting equations 3-58, 62, and 4-41 into equation 4-77 we get,

$$0 = \frac{1}{c^2}\, b_{\alpha\beta} u_\mu \{\xi u^{\dot\alpha} u^\beta u^\mu + 3\zeta \Delta^{(\alpha\mu} u^{\beta)} + \frac{1}{c}\, C_\nu u_\nu\, \{\mu u^\mu u^\nu + P\Delta^{\mu\nu}\} + C_5 u_\mu (\rho u^\mu) \;.$$

Assuming that $g^{\alpha\beta}\, b_{\alpha\beta} = 0$ (20), and defining

$$b_{\alpha\beta} u^\alpha u^\beta \equiv H(x) \quad ,$$

the above equation becomes,

$$(4\text{-}78)\qquad C_\nu u^\nu + \frac{\rho c}{\mu}\, C_5 = -\frac{1}{c\mu}\, (\rho + \frac{4}{c^2}\, \eta\, P) H(x) \quad .$$

From the other fitting condition, equation 4-5, we have

$$(4\text{-}79)\qquad 0 = T_1{}^{\mu\nu} u_\mu u_\nu = \frac{1}{c^2}\, b_{\alpha\beta}\, V_o{}^{\alpha\beta\mu\nu} u_\mu u_\nu + \frac{1}{c}\, C_\alpha\, U_o{}^{\alpha\mu\nu}\, u_\mu u_\nu + $$

$$+ C_5\, T_o{}^{\mu\nu}\, u_\mu u_\nu \quad .$$

Using equations 3-62, 4-41, and the fact that

$$(4\text{-}80)\qquad V_o{}^{\alpha\beta\mu\nu} = 4\pi m\alpha\, \{\frac{K_5}{\beta} - \frac{6K_4}{\beta^2} + \frac{3K_3}{\beta^3}\}\;\; u^\alpha u^\beta u^\mu u^\nu$$

$$+ 4\pi\, m\, \alpha\, c^2\; \{\frac{K_4}{\beta^2} - \frac{K_3}{\beta^3}\}\;\; 4\, u^{(\nu}\Delta^{\mu)(\alpha} u^{\beta)}$$

$$+ 4\pi\ m\,\alpha\,c^2\ \frac{K_3}{\beta^3}\ \{\Delta^{\nu\alpha}\Delta^{\mu\beta} + \Delta^{\nu\beta}\Delta^{\mu\alpha} + \Delta^{\alpha\beta}\Delta^{\mu\nu}\}$$

$$+ 4\pi\ \alpha\ m\ c^2\ \{\frac{K_4}{\beta^2} - \frac{K_3}{\beta^3}\}\ \{u^\alpha u^\beta \Delta^{\mu\nu} + u^\mu u^\nu \Delta^{\alpha\beta}\}\ , \qquad (21)$$

equation 4-79 becomes

(4-81)
$$0 = 4\pi\ m\ c^2\alpha\ \{\frac{K_5}{\beta} - \frac{5K_4}{\beta^2} + \frac{2K_3}{\beta^3}\}\ H(x)$$
$$+ c^3\ \xi\ C_\alpha\ u^\alpha + \mu\ c^4\ C_5 \qquad .$$

Using the recursion relation

$$K_{n+1} = K_{n-1} + \frac{2n}{\beta}\ K_n$$

we see that,

$$K_5 = K_3 + \frac{8}{\beta}\ K_4 \quad , \quad K_4 = K_2 + \frac{6}{\beta}\ K_3\ ,$$

thus

(4-82)
$$K_5 = K_3 + \frac{8K_2}{\beta} + \frac{48}{\beta^2}\ K_3 \quad .$$

From 4-42, 3-60, and equation 4-82, equation 4-81 becomes

(4-83)
$$C_\alpha u^\alpha + \frac{\mu}{\xi}\ c\ C_5 = -\ \frac{1}{c\xi}\ \ \{(\beta + \frac{20}{\beta})\ \frac{\zeta}{c^2} + \frac{3P}{c^2}\}\ H(x) \quad .$$

From equations 4-78 and 4-83 we are thus able to solve for $C_\alpha u^\alpha$ and C_5 :

$$(4\text{-}84) \qquad C_\alpha u^\alpha = \left\{ - \frac{\mu c^3(1-\gamma)(\xi + \frac{\zeta}{c^2})}{P^2} + \frac{\rho c(1-\gamma)[(\beta + \frac{20}{\beta})\zeta + 3P]}{P^2} \right\} H(x) ,$$

and

$$(4\text{-}85) \qquad C_5 = \left\{ - \frac{\mu(1-\gamma)[(\beta + \frac{20}{\beta})\zeta + 3P]}{P^2} + \frac{\xi c^2(\xi + \frac{\zeta}{c^2})(1-\gamma)}{P^2} \right\} H(x) ,$$

where we have also used the fact that

$$(4\text{-}86) \qquad \mu^2 - \rho \xi = \frac{P^2}{c^4(1-\gamma)} . \quad (22)$$

Thus we see that in equations 4-74, 75 we have $M_1{}^\mu$, and $T_1{}^{\mu\nu}$ expressed in terms of $b_{\alpha\beta}$. Our next task is to express the Heat flow vector, and the Viscous Strees Tensor in terms of $b_{\alpha\beta}$; we do this by means of equations 4-22, 23. Let us first consider the heat flow vector, Q^μ . From equations 4-74, 3-58, 3-62, 4-41, the first term on the right hand side of equation 4-22 becomes

$$(4\text{-}87) \qquad - \eta c^2 M_1{}^\mu = - \eta \xi H(x) u^\mu - 2\eta \zeta b_{\alpha\beta} \Delta^{\alpha\mu} u^\beta - \frac{\eta\zeta}{c^2} H(x) u^\mu -$$

$$- \eta \mu c (C_\nu u^\nu) u^\mu - P \eta c C_\nu \Delta^{\mu\nu} - \eta \rho c^2 C_5 u^\mu .$$

Upon substituting equations 4-84, 85 into equation 4-87 we get

$$- \eta\, c^2\, M_1{}^{\mu} = \{- \eta\, \xi - \frac{\eta\zeta}{c^2} + \eta\, c^4\, \frac{(1-\gamma)(\xi + \frac{\zeta}{c^2})}{P^2}\, (\mu^2 - \rho\xi)\}\; H(x)\; u^{\mu}$$

$$- 2\,\zeta\, b_{\alpha\beta}\Delta^{\alpha\mu}\, u^{\beta} - P\,\eta\, c\, C_{\nu}\, \Delta^{\mu\nu} \quad .$$

By means of equation 4-86, the above simplifies to

(4-88) $$- \eta\, c^2\, M_1{}^{\mu} = - 2\,\eta\xi\, b_{\alpha\beta}\Delta^{\alpha\mu}\, u^{\beta} - P\,\eta\, c\, C_{\nu}\Delta^{\mu\nu} \quad .$$

Similarly, by substituting equations 3-62, 4-41, 80 into equation 4-74, the second terms of equation 4-22 becomes

(4-89) $$- \Delta^{\mu}{}_{\rho}\, T_1{}^{\rho\nu}\, u_{\nu} = 4\pi\, m\, \alpha\, c^2 (\frac{K_4}{\beta^2} - \frac{K_3}{\beta^3})(2)\, b_{\alpha\beta}\, \Delta^{\mu\alpha} u^{\beta} + \zeta\, c\, C_{\alpha}\, \Delta^{\mu\alpha} \; .$$

Thus by adding equations 4-87, and 4-89, equation 4-22 becomes

(4-90) $$Q^{\mu} = - 2\, P\, [\eta^2 - \frac{5\eta}{\beta} - 1] b_{\alpha\beta}\, \Delta^{\alpha\mu}\, u^{\beta}$$

and using the fact that

$$- \frac{1}{\beta^2}\, (\frac{\gamma}{\gamma-1}) = \eta^2 - \frac{5}{\beta}\,\eta - 1 \quad (23)$$

equation 4-97 finally yields

(4-91) $$Q^{\mu} = \frac{2P}{\beta^2}\, (\frac{\gamma}{\gamma-1}) b_{\alpha\beta}\, \Delta^{\alpha\mu}\, u^{\beta} \quad .$$

In order to replace the quantity $b_{\alpha\beta}\,\Delta^{\alpha\mu}\,u^{\beta}$ by gradients of thermodynamic variables we have to go back to equation 4-39, and make use of equations 4-61, 62, 63, 71, i.e.

$$\eta\, P\, \sigma^{\lambda\mu} + q^{\mu}u^{\lambda} + q^{\lambda}u^{\mu} + A_2 u^{\alpha}{}_{|\alpha}\, u^{\lambda}u^{\mu} + A_3\, u^{\alpha}{}_{|\alpha}\, \Delta^{\lambda\mu} = \tag{4-92}$$

$$= -b_{\alpha\beta}\,\{A\, u^{\lambda}u^{\mu}u^{\alpha}u^{\beta} + B(\Delta^{\lambda\alpha}\Delta^{\mu\beta} + \Delta^{\mu\alpha}\Delta^{\lambda\beta}) + C(u^{\lambda}u^{\mu\alpha}u^{\beta} +$$

$$+ u^{\mu}\Delta^{\lambda\alpha}u^{\beta} + u^{\lambda}\Delta^{\mu\beta}u^{\alpha} + u^{\mu}\Delta^{\lambda\mu}u^{\alpha}) + D\, \Delta^{\lambda\mu}\,\Delta^{\alpha\beta} \qquad +$$

$$+ E(\Delta^{\lambda\mu}u^{\alpha}u^{\beta} + \Delta^{\alpha\beta}u^{\lambda}u^{\mu})\} \quad .$$

Multiplying equation 4-92 by $u_{\lambda}u_{\mu}$ we get,

$$c^4\, A_2\, u^{\alpha}{}_{|\alpha} = -\, c^4\, A\, b_{\alpha\beta}\, u^{\alpha}u^{\beta} - E\, c^2\, b_{\alpha\beta}\, u^{\alpha}u^{\beta} \quad , \tag{4-93}$$

where we have used the fact that $u_{\mu}\sigma^{\mu\lambda} = u_{\mu}q^{\mu} = 0$, (this follows easily from definition of $\sigma^{\mu\lambda}$, and q^{μ}). But in view of equation 4-72, equation 4-93 yields the result,

$$A_2 = -\,\frac{4}{3}\,\frac{A}{u^{\alpha}{}_{|\alpha}}\; H(x) \; . \tag{4-94}$$

Multiplying equation 4-92 by $\Delta_{\lambda\mu}$ and noting the fact that $\sigma^{\lambda}_{\lambda} = 0$, we get

(4-95) $$3 A_3 u^{\alpha}{}_{|\alpha} = - \left(\frac{2B}{c^2} + \frac{3D}{c^2}\right) H(x) - 3E\, H(x) \quad .$$

But in view of equation 4-73, equation 4-95 simplifies to,

(4-96) $$A_3 = - \frac{4}{9} \frac{A\, c^2}{u^{\alpha}{}_{|\alpha}} \quad .$$

Multiplying equation 4-92 by u_{μ} , and noting equation 4-94, we get

(4-97) $$q^{\lambda} = - 2C \, (b_{\alpha\beta} \, \Delta^{\lambda\alpha} \, u^{\beta}) \quad .$$

Thus substituting 4-97 into equation 4-91, and noting the definition of q^{μ} (i.e. equation 4-62) we finally get

(4-98) $$Q^{\mu} = \frac{1}{\eta C} \left[\frac{P}{\beta^2} \left(\frac{\gamma}{\gamma-1}\right)\right]^2 \Delta^{\mu}{}_{\alpha} \, \theta^{|\alpha}$$

thus the thermal conductivity λ is

(4-99) $$\lambda = \frac{1}{\eta C} \left[\frac{P}{\beta^2} \left(\frac{\gamma}{\gamma-1}\right)\right]^2 \quad .$$

We still have to calculate C as defined in equation 4-65. This will be done at a later stage.

Let us next consider the viscous stress tensor. From equations 4-34, 75, 80, 41, and 3-62 we have

$$\tau_{\alpha\beta} = \frac{1}{c^2} b_{\gamma\delta} \{4\pi\, m\alpha c^4 \frac{K_3}{\beta^3} (2\Delta^\gamma{}_\alpha \Delta^\delta{}_\beta + \Delta^{\gamma\delta}\Delta_{\alpha\beta}) \tag{4-100}$$

$$+ 4\pi\, m\alpha c^2 (\frac{K_4}{\beta^2} - \frac{K_3}{\beta^3})\, u^\gamma u^\delta \Delta_{\alpha\beta} \quad + \frac{1}{c} \zeta\, C_\gamma u^\gamma)\, \Delta_{\alpha\beta}$$

$$+ C_5\, P\, \Delta_{\alpha\beta} \quad .$$

In view of equations 3-60, 4-42, 82, equation 4-100 becomes

$$\tau_{\alpha\beta} = \frac{\eta P}{\beta} (2\, b_{\gamma\delta}\Delta^\gamma{}_\alpha \Delta^\delta{}_\beta) + (\frac{\eta\, P}{c^2\beta} H + \frac{1}{c^2} (P + \frac{5\eta P}{\beta})\, H + \tag{4-101}$$

$$+ \frac{1}{c} \zeta\, C_\gamma u^\gamma + C_5 P)\, \Delta_{\alpha\beta}.$$

In order to eliminate the quantity $b_{\gamma\delta}\Delta^\delta{}_\alpha\Delta^\delta{}_\beta$ we once again go back to equation 4-92. Substituting equations 4-94, 95, 96, 103, 104 into equation 4-91, and noting equations 4-72, 73, 81 we have

$$\eta\, P\, \sigma^{\lambda\mu} - \frac{2}{3} \frac{B}{c^2} H(x)\, \Delta^{\lambda\mu} = -\, 2\, B\, b_{\alpha\beta}\Delta^{\lambda\alpha}\Delta^{\mu\beta} \quad . \tag{4-102}$$

Making use of equation 4-102, equation 4-101 becomes

$$\tau_{\alpha\beta} = -\, \frac{(\eta P)^2}{\beta\, B}\, \sigma_{\alpha\beta} + [\frac{5}{3} \frac{\eta\, P}{\beta c^2} H + \frac{1}{c^2} (P + \frac{5\eta P}{\beta})\, H \tag{4-103}$$

$$+ \frac{1}{c} \zeta\, (C_\gamma u^\gamma) + C_5 P]\, \Delta_{\alpha\beta} \quad .$$

Let us now calculate the coefficient of $\Delta_{\alpha\beta}$: Upon substituting in the values of $C_\gamma u^\gamma$ and C_5 from equations 4-84, 85, 93 we get;

$$\{\frac{20}{3}\frac{\eta}{\beta}\frac{P}{c^2} + \frac{P}{c^2} + [c^2(P\xi - \mu\zeta)(\xi + \frac{\zeta}{c^2}) + (\rho\zeta - \mu P)[(\beta + \frac{20}{\beta})\zeta + 3P]] \times \frac{(1-\gamma)}{P^2}\}$$

Using the identities (24) $\rho\zeta - \mu P = \frac{P^2}{c^2}$, and $\mu\zeta - \xi P = \frac{P^2}{\beta c^2}[\eta\beta - \frac{\gamma}{\gamma-1}]$ the above simplifies to

$$\{\frac{20}{3}\frac{\eta P}{\beta c^2} - \frac{4P}{c^2}(\eta^2 - \frac{5\eta}{\beta} - 1) + \frac{4\gamma P}{c^2}(\eta^2 - \frac{5\eta}{\beta} - 1) - \frac{4\eta\gamma P}{\beta c^2}\} \; H(x) \; .$$

Making use of equation 4-55 the above yields

$$(4\text{-}104) \qquad -\frac{4P\,\eta}{3c^2\beta}[3\gamma - 5 + \frac{3\gamma}{\beta\eta}] \; H(x) \; .$$

But by equation 4-94, 4-104 becomes

$$(4\text{-}105) \qquad -\frac{1}{A}\left(\frac{P\eta}{c^2}\right)^2 \frac{1}{\beta}\,\Omega^2\, u^\alpha{}_{|\alpha}$$

where $\Omega = (3\gamma - 5 + \frac{3\gamma}{\beta\eta})$. Expression 4-105 is the coefficient of $\Delta_{\alpha\beta}$ in equation 4-103. Therefore the viscous stress tensor becomes,

$$(4\text{-}106) \qquad \tau_{\alpha\beta} = -\left[\frac{(\eta P)^2}{\beta B}\right]\sigma_{\alpha\beta} - \left[\frac{1}{A}\left(\frac{P\eta}{c^2}\right)^2 \frac{1}{\beta}\,\Omega^2\right] u^\lambda{}_{|\lambda}\,\Delta_{\alpha\beta} \; .$$

But this is just the phenomenological representation of the viscous stress tensor (see equation 4-17). Thus we are able to identify the shear viscosity and bulk viscosity with the coefficients of $\sigma_{\alpha\beta}$,

and $u^{\lambda}{}_{|\lambda}\,\Delta_{\alpha\beta}$ respectively, i.e.

$$\nu = \frac{(\eta P)^2}{\beta B} \quad , \quad \kappa = \left(\frac{P\eta\Omega}{c}\right)^2 \frac{1}{A\beta} \quad . \tag{4-107}$$

As was mentioned in the case of conductivity the quantities A , B , C have to be calculated before we have the coefficients just as functions of temperature T (or β), and γ . This next section will involve choosing proper coordinates, basis vectors, and then integrating to get A , B , C .

Integrations

Preliminaries.

In order to perform the integrations we choose proper coordinates with the time axis along u_μ , so that $u_\mu = (0,0,0,-c)$. In order to specify the relative momentum of the incoming particles we choose an orthonormal triad $\underset{\sim}{e}_1$, $\underset{\sim}{e}_2$, and $\underset{\sim}{e}_3$ in the centroidal 3-flat. The components of the vectors $\underset{\sim}{e}_1$, $\underset{\sim}{e}_2$, $\underset{\sim}{e}_3$ are, (see figure 1)

$$e^{\alpha}_{(3)} = (\cosh\bar{\chi}\,\sin\theta\,\cos\phi,\ \cosh\bar{\chi}\,\sin\theta\,\sin\phi,\ \cosh\bar{\chi}\,\cos\theta,\ \sinh\bar{\chi}) \quad , \tag{4-108}$$

$$e^{\alpha}_{(2)} = (\sin\phi,\ \cos\phi,\ 0,0) \quad ,$$

$$e^{\alpha}_{(1)} = (\sin\theta\,\cos\phi,\ \sin\theta\,\sin\phi,\ \cos\theta,\ 0) \quad .$$

It can easily be seen that $\underset{\approx}{e}_1$, $\underset{\approx}{e}_2$, $\underset{\approx}{e}_3$ do indeed form an orthonormal triad, i.e.

$$\underset{\approx}{e}_{(a)} \cdot \underset{\sim}{e}_{(b)} = \gamma_{\alpha\beta}\, e^{\alpha}_{(a)}\, e^{\beta}_{(b)} = \delta_{ab} \quad , \tag{4-109}$$

where $\gamma_{\alpha\beta}$ is the Minkowski metric, i.e. diag. (1,1,1,-1) , and $\alpha,\beta, = 1,2,3,4$, and $a,b, = 1,2,3$.

It can also be shown that $\underset{\approx}{e}_1, \underset{\approx}{e}_2, \underset{\approx}{e}_3$ due in fact lie in the centroidal 3-flat;

$$\underset{\approx}{\tilde{e}}_{(a)} \cdot \underset{\sim}{\overline{P}} = \gamma_{\alpha\beta}\, e^{\alpha}_{(a)}\, \overline{P}^{\beta} = 0 \quad \forall_a$$

e.q. consider the case for $a = 3$, we know that

$$\overline{P}^{\alpha} = m\, c(\sinh \overline{\chi} \sin \theta \cos \phi, \sinh \overline{\chi} \sin \theta \sin \phi, \sinh \overline{\chi} \cos \theta, \cosh \overline{\chi}]$$

thus

$$\begin{aligned} \underset{\approx}{e}_3 \cdot \underset{\sim}{\overline{P}} = \gamma_{\alpha\beta}\, e^{\alpha}_{(3)}\, \overline{P}^{\beta} &= m\, c(\sinh \overline{\chi} \cosh \overline{\chi} \sin^2\theta \cos^2\phi + \\ &+ \sinh \overline{\chi} \cosh \overline{\chi} \sin^2\theta \sin^2\phi + \\ &+ \sinh \overline{\chi} \cosh \overline{\chi} \cos^2\theta - \cosh \overline{\chi} \times \\ &\times \sinh \overline{\chi}) = m\, c\, \{\sinh \overline{\chi} \cosh \overline{\chi} \times \end{aligned}$$

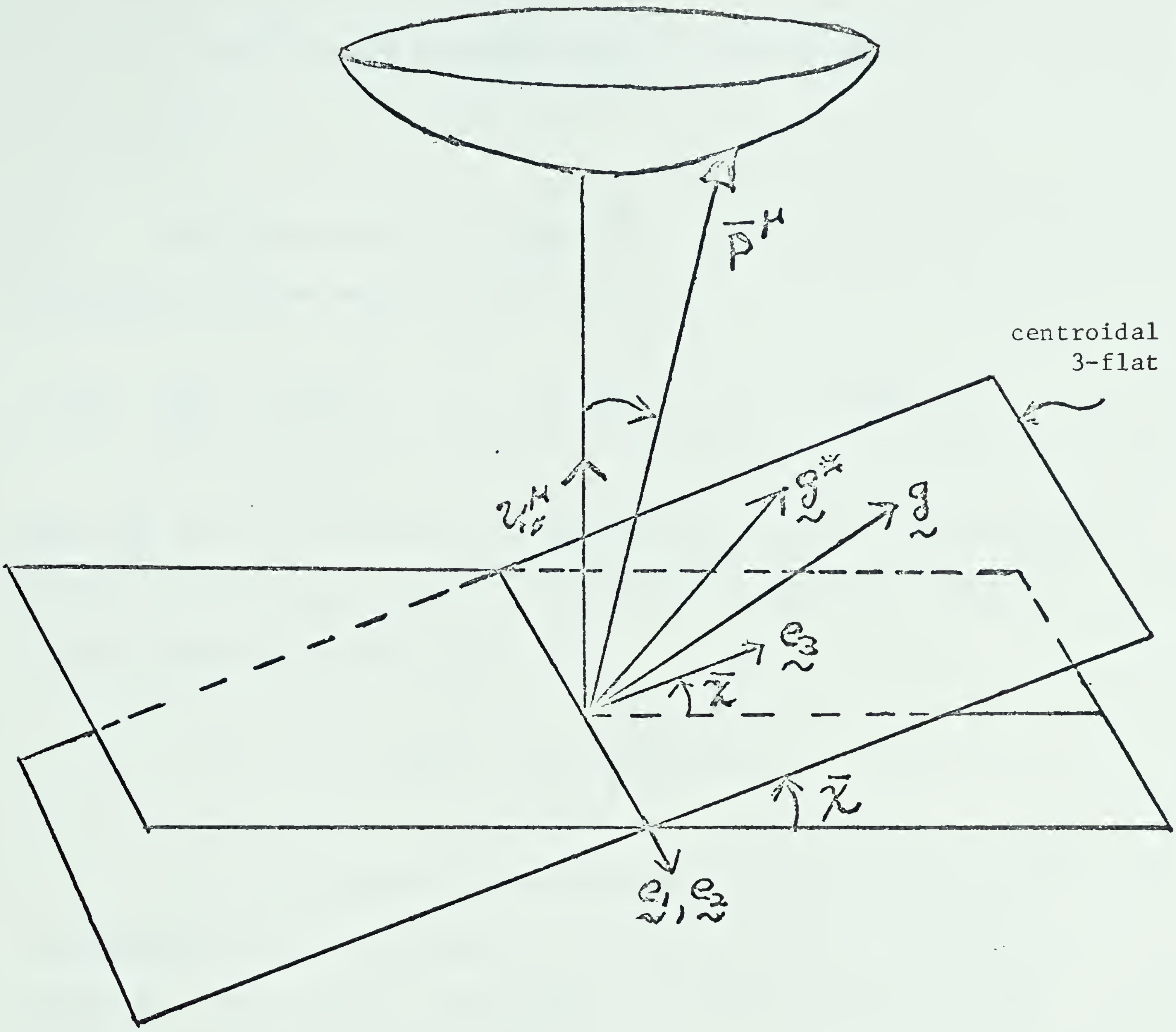


FIGURE 1

$$\times\ [\sin^2\theta\ (\cos^2\phi + \sin^2\phi) + \cos^2\theta] - \cosh\bar{\chi}\ \sinh\bar{\chi}\}$$

$$= 0\quad .$$

Thus using polar coordinates ϑ , and ϵ we can specify the normalized relative momentum vector g^{α}/g by

$$\text{(4-110)}\qquad \frac{g^{\alpha}}{g} = \sin\vartheta\cos\epsilon\ e^{\alpha}_{(1)} + \sin\vartheta\sin\epsilon\ e^{\alpha}_{(2)} + \cos\vartheta\ \cos^{\alpha}_{(3)}$$

where ϑ and ϵ are measured as indicated by figure 2. Using equation 4-110, it can be shown that g^{α}/g as defined by equation 4-110 does indeed satisfy the condition,

$$\frac{g^{\alpha}}{g}\cdot\frac{g_{\alpha}}{g} = \gamma_{\alpha\beta}\frac{g^{\alpha}}{g}\frac{g^{\beta}}{g} = 1\quad .$$

In order to specify the normalized relative momentum of the outgoing particles, i.e. $\underset{\sim}{g_*}/g$, we make use of another orthonormal triad $\underset{\sim}{E}_1$, $\underset{\sim}{E}_2$, and $\underset{\sim}{E}_3$, (see figure 3), in the centroidal 3-flat. The vectors in the triad are defined as

$$E^{\alpha}_{(3)} = \frac{g^{\alpha}}{g}\quad ,$$

$$E^{\alpha}_{(2)} = -\sin\epsilon\ e^{\alpha}_{(1)} + \cos\epsilon\ e^{\alpha}_{(2)}\quad ,$$

$$E^{\alpha}_{(3)} = \cos\vartheta\cos\epsilon\ e^{\alpha}_{(1)} + \cos\vartheta\sin\epsilon\ e^{\alpha}_{(2)} - \sin\vartheta\ e^{\alpha}_{(3)}\quad ,$$

where ϑ , ϵ are those values that specify g^{α}/g .

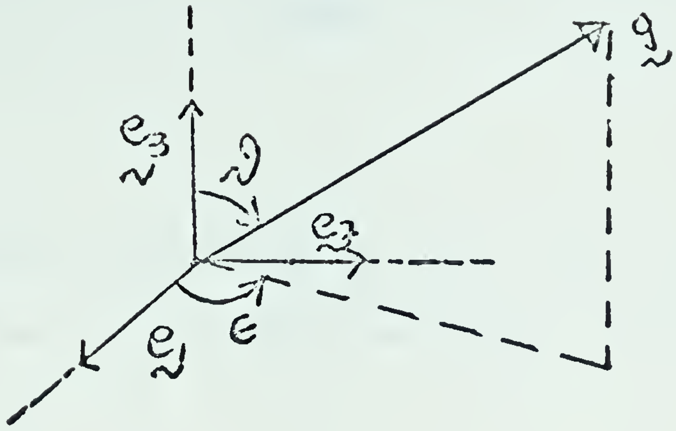


FIGURE 2

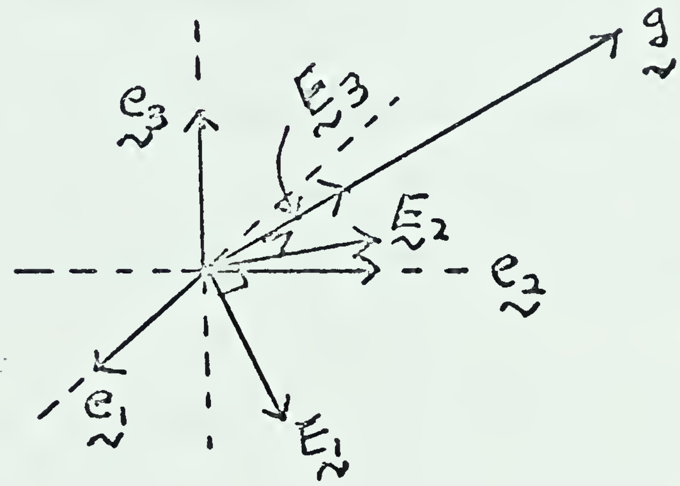


FIGURE 3

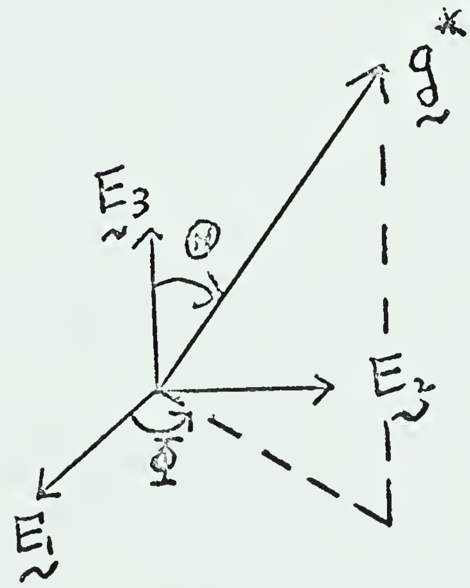


FIGURE 4

Thus relative to the triad $\underset{\sim}{E}_1$, $\underset{\sim}{E}_2$, and $\underset{\sim}{E}_3$, the normalized relative momentum of the outgoing particles is given by

$$(4\text{-}112) \qquad \frac{\underset{\sim}{g}_*}{g} = \sin\Theta \cos\Phi \underset{\sim}{E}_1 + \sin\Theta \sin\Phi \underset{\sim}{E}_2 + \cos\Theta \underset{\sim}{E}_3$$

where Θ , and Φ are measured as indicated by figure 4.

Now that some of the preliminaries have been discussed let us now calculate the coefficients A , B and C . Let us first consider A . From equation 4-64 and the definition of $X^{\alpha\beta\lambda\mu}$ we see that

$$A = \frac{1}{4} mc^{-5} \iiiint W N_o \, 'N_o \, u_\alpha u_\beta \delta\{P^\alpha P^\beta\} \, u_\lambda u_\mu \, \delta\{P^\lambda P^\mu\} d\omega \, d'\omega \, d\omega^* \, d'\omega^* \quad ,$$

which becomes

$$(4\text{-}113) \qquad A = \frac{1}{4} mc^{-5} \iiiint W N_o \, 'N_o \, \{\delta[\underset{\sim}{P}^2]\}^2 \, d\omega \, d'\omega \, d\omega^* \, d'\omega^*$$

where

$$(4\text{-}114) \qquad \delta[\underset{\sim}{P}^2] \equiv [(u_\mu P^\mu)^2 + (u_\mu 'P^\mu)^2 - (u_\mu P^{*\mu})^2 - (u_\mu 'P^{*\mu})^2] \quad .$$

But we know from equations 2-25, 27 that

$$P_\mu = (1 + \frac{1}{4} g^2) \, \overline{P}_\mu - \frac{1}{\alpha} g_\mu$$

$$P^*_\mu = (1 + \frac{1}{4} g^2)^{\frac{1}{2}} \bar{P}^*_\mu - \frac{1}{2} g^*_\mu$$

$$'P_\mu = (1 + \frac{1}{4} g^2)^{\frac{1}{2}} \bar{P}_\mu + \frac{1}{2} g^*_\mu$$

(4-115) $$'P^*_\mu = (1 + \frac{1}{4} g^2)^{\frac{1}{2}} \bar{P}^*_\mu + \frac{1}{2} g^*_\mu \quad .$$

Substituting equation 4-11 into equation 4-113, and using the fact that $\bar{P}_\mu = \bar{P}^*_\mu$, we get

$$\delta[\mathbb{P}^2] = \frac{1}{2} [(u_\mu g^\mu)^2 - (g^{*\mu} u_\mu)^2] \quad .$$

But since we have $u_\mu = (0,0,0,-c)$, we finally get

(4-116) $$\delta[\mathbb{P}^2] = \frac{1}{2} c^2 [(g^4)^2 - (g^{*4})^2] \quad .$$

But inview of the preliminaries just discussed we know that

(4-117) $$g^4 = g\, e^4_{(3)} \cos\vartheta = g \cos\vartheta \sinh \bar{\chi} \quad ,$$

$$g^{*4} = -g \sin\vartheta\, e^4_{(3)} \sin\Theta \cos\Phi + g\, e^4_{(3)} \cos\vartheta \cos\Theta$$

(4-118) $$g^{*4} = g \sinh \bar{\chi} (\cos\vartheta \cos\Theta - \sin\vartheta \sin\Theta \cos\Phi) \quad .$$

Thus from equations 4-116, 117, 118, 125, 126 we can now write

(4-119) $$[\delta\{\mathbb{P}^2\}]^2 = \frac{1}{4} c^4 (g)^4 \sinh^4 \bar{\chi} \{\cos^2\vartheta \sin^4\Theta + \sin^4\vartheta \sin^4\Theta \cos^4\Phi$$

$$+ 4 \sin^2\vartheta \cos^2\vartheta \sin^2\Theta \cos^2\Theta \cos^2\Phi - 2 \sin^2\vartheta \cos^2\vartheta \sin^4\Theta \cos^2\Phi$$

$$- 4 \sin^3\vartheta \cos\vartheta \sin^3\Theta \cos\Theta \cos^3\Phi + 4 \sin\vartheta \cos^3\vartheta \sin^3\Theta \cos\Theta \cos\Phi \} .$$

Before performing the integrations, let us consider the other terms in equation 4-113. The term $N_o\,'N_o$ becomes

$$N_o\,'N_o = \alpha^2 e^{\frac{2\beta}{c}(P^\mu + 'P^\mu)u_\mu} = \alpha^2 \exp\left[\frac{2\beta}{c}\left(1 + \frac{1}{4} g^2\right)^{\frac{1}{2}} u_\mu \overline{P}^\mu\right] ,$$

which simplifies to

$$(4\text{-}120) \qquad N_o\,'N_o = \alpha^2 e^{-2\beta\left(1 + \frac{1}{4} g^2\right)^{1/2} \cosh \overline{\chi}} .$$

Making use of equations 2-37, 38, we have

$$d\omega\, d'\omega = \left(1 + \frac{1}{4} g^2\right)^{\frac{1}{2}} g^2\, d\overline{\omega}\, dg \sin\vartheta\, d\vartheta\, d\epsilon$$

$$(4\text{-}121) \qquad d\omega\, d'\omega = \left(1 + \frac{1}{4} g^2\right)^{\frac{1}{2}} g^2 \sin\vartheta \sinh^2\overline{\chi} \sin\overline{\theta}\, d g\, d\overline{\chi}\, d\overline{\theta} d\overline{\phi} d\vartheta d\epsilon$$

and

$$(4\text{-}122) \qquad d\omega^*\, d'\omega^* = \left(1 + \frac{1}{4}(g^*)^2\right)^{\frac{1}{2}} (g^*)^2\, d\overline{\omega}^*\, dg^* \sin\Theta\, d\Theta\, d\Phi .$$

From equation 2-35 we know that

$$W(P,'P:P^*'P^*) = \left(1 + \frac{1}{4} g^2\right)^{-\frac{1}{2}} g^{-1} \sigma(g,\Theta)\delta_\omega(\overline{P}^*_\mu - \overline{P}_\mu)\delta(g^* - g) .$$

Thus making use of equations 4-120, 121, 122 and equation 2-35, integrating over $d\bar{\omega}^* \; dg^*$, and making use of the delta functions, equation 4-113 becomes

$$A = \frac{1}{4} m \alpha^2 c^{-5} \int \cdots \int \sigma(g,\Theta) \exp [-2\beta(1 + \frac{1}{4} g^2)^{\frac{1}{2}} \cosh \bar{\chi}] \times \tag{4-123}$$

$$\times [\delta\{P^2\}]^2 g^3 (1 + \frac{1}{4} g^2)^{\frac{1}{2}} \sinh^2 \bar{\chi} \sin \bar{\theta} \sin \vartheta \sin \Theta \quad \times$$

$$\times\, dg\, d\bar{\chi}\, d\bar{\theta}\, d\bar{\phi}\, d\vartheta\, d\epsilon\, d\Theta\, d\Phi \quad .$$

Substituting equation 4-121 into equation 4-123, and integrating over $\bar{\theta}, \bar{\phi}, \epsilon, \vartheta$ and Φ we obtain

$$A = \frac{16}{5} \pi^3 \alpha^2 m c^{-1} \int \cdots \int \sinh^6 \bar{\chi} \exp [-2\beta(1 + \frac{1}{4} g^2)^{\frac{1}{2}} \cosh \bar{\chi}] \times$$

$$\times (\frac{2}{3} \sin^5\Theta + \frac{2}{3} \sin^3\Theta \cos^2\Theta) (1 + \frac{1}{4} g^2)^{\frac{1}{2}} g^7 \sigma(g,\Theta) dg\, d\Theta\, d\bar{\chi} \quad .$$

Letting $y \equiv 2\beta(1 + \frac{1}{4} g^2)^{1/2}$ the above becomes

$$A = \frac{16}{5} \pi^3 \alpha^2 c^{-1} \iiint e^{-y \cosh \bar{\chi}} \sinh^6 \bar{\chi} (\frac{y}{2\beta}) g^7 \times \tag{4-124}$$

$$\times\, \sigma(\frac{2}{3} \sin^3\Theta + \frac{2}{3} \sin^3\Theta \cos^2\Theta)\, dg\, d\Theta\, d\bar{\chi} \quad .$$

Using equation 3-55, equation 4-124 becomes on integrating over $\bar{\chi}$,

$$A = \frac{8\pi^3 m\alpha^2 c^{-1}}{\beta} \iint \sin^3\Theta \frac{K_3(y)}{y^2} g^7 \sigma(g,\Theta)\, dg\, d\Theta \quad . \tag{4-125}$$

Let us next consider the coefficient B (see equation 4-69). There are two integrations that have to be performed. Breaking up the calculation of B into two different integrations we have,

(4-126)
$$F = \frac{1}{10} \Delta_{\alpha\lambda} \Delta_{\beta\mu} X^{\alpha\beta\lambda\mu}$$
$$G = \frac{1}{10} (- \frac{1}{3} \Delta_{\alpha\beta} \Delta_{\lambda\mu}) X^{\alpha\beta\lambda\mu} \quad .$$

We have essentially already calculated G in our calculation of A . Using identity

$$\Delta_{\mu\nu} P^{\mu}P^{\nu} = \frac{(u_{\mu}P^{\mu})^2}{c^2} - 1 \equiv \frac{\mathcal{P}^2}{c^2} - 1$$

we see that

$$\Delta_{\alpha\beta}\Delta_{\lambda\mu} X^{\alpha\beta\lambda\mu} = \frac{1}{4} m c^3 \{\frac{1}{c^4} \int\cdots\int W N_o 'N_o [\delta\{\mathcal{P}^2\}]^2 d\omega d'\omega d\omega^* d'\omega^* \quad ,$$

since

$$\delta[\frac{(u_{\mu}P^{\mu})^2}{c^2} - 1] = \delta[\frac{(u_{\mu}P^{\mu})^2}{c^2}] - \delta[1] = \delta[\frac{(u_{\mu}P^{\mu})^2}{c^2}] \quad .$$

But we've already calculated the preceding integral, so that it can easily be seen that

(4-127)
$$\Delta_{\alpha\beta}\Delta_{\lambda\mu} X^{\alpha\beta\lambda\mu} = \frac{8\pi^3 m\alpha^2 c^3}{\beta} \iint \sin^3\Theta \frac{K_3(y)}{y^2} g^7 \sigma(g,\Theta) dg d\Theta \quad .$$

In calculating F we have to calculate

$$\Delta_{\alpha\lambda}\Delta_{\beta\mu} X^{\alpha\beta\lambda\mu} = \frac{1}{4} m c^3 \int\cdots\int W N_o \,'N_o \Delta_{\alpha\lambda}\Delta_{\beta\mu} \delta\{P^\alpha P^\beta\} \delta\{P^\lambda P^\mu\} \, d\omega \, d'\omega \, d\omega^* \, d'\omega^* . \tag{4-128}$$

But since in proper coordinates $\Delta_{\alpha\lambda}$, and $\Delta_{\beta\mu}$ have components (1,1,1,0) then 4-128 becomes

$$\Delta_{\alpha\lambda}\Delta_{\beta\mu} X^{\alpha\beta\lambda\mu} = \frac{1}{4} m c^3 \int\cdots\int W N_o \,'N_o \delta\{P_a P_b\} \delta\{P^a P^b\} \, d\omega \, d'\omega \, d\omega^* \, d'\omega^* , \tag{4-129}$$

where $a,b = 1,2,3$.

Using equation 4-115 and the fact that $\overline{P}^a = \overline{P}^b$, and $\overline{P}_a = \overline{P}_b$ we can write

$$\begin{aligned} \delta[P^a P^b] &= \frac{1}{2} [g^a g^b - g^{*a} g^{*b}] \\ \delta[P_a P_b] &= \frac{1}{2} [g_a g_b - g^*_a g^*_b] . \end{aligned} \tag{4-130}$$

Thus we can write

$$\delta[P_a P_b]\delta[P^a P^b] = \frac{1}{4} \{(g_a g^a) + (g^*_a g^{*a}) - 2(g_a g^{*a})^2\} . \tag{4-131}$$

Since $g^\alpha g_\alpha = g^{*\alpha} g^*_\alpha = g^2$ we can write

$$g^a g_a = \{g^2 + (g^4)^2\}$$

(4-132)

$$g^{*a} g^*_a = \{g^2 + (g^{*4})^2\} \quad .$$

It can also be easily seen that

$$g^{*\alpha} g_\alpha = g\, g^{*\alpha}\, E_{(3)\alpha} = g^2 \cos\Theta \quad .$$

Thus we can write

(4-133)
$$g^{*a} g_a = g^2 \cos\Theta + g^{*4}\, g^4 \quad .$$

Making use of coordinate system we find that

$$g^4 = g \sinh\bar{\chi} \cos\vartheta$$

$$g^{*4} = -g \sinh\bar{\chi} \sin\vartheta \sin\Theta \cos\Phi + g \sinh\bar{\chi} \cos\vartheta \cos\Theta$$

$$g^a g_a = g^2 (1 + \sinh^2\bar{\chi} \cos^2\vartheta)$$

(4-134)

$$g^{*a} g^*_a = g^2 \{1 + \sinh^2\bar{\chi} [\sin^2\vartheta \sin^2\Theta \cos^2\Phi + \cos^2\vartheta \cos^2\Theta - 2 \sin\vartheta \cos\vartheta \sin\Theta \cos\Theta \cos\Phi]\}$$

$$g^{*a} g_a = g^2 \{\cos\Theta + \sinh^2\bar{\chi} [\cos^2\vartheta \cos\Theta - \sin\vartheta \cos\vartheta \sin\Theta \cos\Phi]\}$$

Setting up our integrations in exactly the same fashion as we did in calculating, and after many integrations over $\bar{\chi}, \bar{\theta}, \bar{\phi}$, $, \epsilon, \vartheta$ and Φ we finally arrive at

$$(4\text{-}135)\quad \Delta_{\alpha\lambda}\Delta_{\beta\mu}\, X^{\alpha\beta\lambda\mu} = \frac{8\pi^3 m\alpha^2 c^3}{\beta} \iint \left(K_1(y) + \frac{2K_2}{y} + \frac{3K_3}{y^2}\right) \sin^3\Theta\; g^7$$

$$\sigma(g,\Theta)\; dg\; d\Theta \quad ,$$

where in integrating over $\overline{\chi}$, we once again let $y \equiv 2\beta(1 + \frac{1}{4} g^2)^{1/2}$ and made use of equation 3-55. Thus from equations 4-120, 121, 129 the coefficient B (see equation 4-69) becomes

$$(4\text{-}136)\quad B = \frac{4}{5} \frac{\pi^3 m\alpha^2 c^3}{\beta} \int\cdots\int \left(K_1(y) + \frac{2K_2(y)}{y} + \frac{4}{3} \frac{K_3(y)}{y^2}\right) \sin^3\Theta \quad g^7$$

$$\sigma(g,\Theta)\quad dg\; d\Theta \; .$$

From equations 4-65, 39, the coefficient C is

$$(4\text{-}137)\quad C = \frac{1}{12}\, m\, c^{-1} \int\cdots\int W\, N_o\, {}'N_o\; u_\lambda \Delta_{\mu\alpha} u_\beta\; \delta\{P^\alpha P^\beta\}\delta\{P^\lambda P^\mu\} d\omega\; d'\omega\; d\omega^*\; d'\omega^* \; .$$

In proper coordinates equation 4-137 reduces to

$$(4\text{-}138)\quad C = \frac{1}{12}\, m\, c \int\cdots\int W\, N_o\, {}'N_o\; [\delta\{P^4 P_a\}]\quad [\delta\{P^4 P^a\}]\; d\omega\; d'\omega\; d\omega^*\; d'\omega^* \quad ,$$

where $a = 1,2,3$.

From equation 4-130 it follows that

$$(4\text{-}139)\quad \begin{aligned} \delta\{P^4 P_a\} &= \frac{1}{2}\,(g^4 g_a - g^{*4} g^*{}_a) \\ \delta\{P^4 P^a\} &= \frac{1}{2}\,(g^4 g^a - g^{*4} g^{*a}) \quad . \end{aligned}$$

Making use of equation 4-139, 134 and once again embarking upon many integrations we finally arrive at,

$$C = \frac{2}{3} \pi^3 \alpha^2 m c \iint \left(\frac{K_2(y)}{y} + \frac{2K_3}{y^2}\right) \sin^3\Theta \, g^7 \, \sigma(g,\Theta) \, dg \, d\Theta \quad . \tag{4-140}$$

CHAPTER 5

This concluding chapter will be devoted to a brief discussion of the results, the limitations, and the possible extensions of the theory of the previous chapters.

If we compute the asymptotic behavior of the bulk viscosity, using the necessary expressions given in chapter 4 we find that for the classical $(\beta\to\infty)$ and the ultrarelativistic $(\beta\to 0)$ cases the bulk viscosity vanishes for a simple gas (1). We also see why it is impossible to have an isentropic expansion of a relativistic gas (2).

As an application of the above let us consider the problem of a spherically symmetric expanding universe filled with matter, i.e. the Robertson-Walker Universe. In the usual discussions that were given about the Robertson-Walker Universe, we had a universe filled with some fluid that had no shearing or heat conduction, and that is expanding. This would imply that it is expanding with a constant entropy (i.e. no dissipative agents are present). But from our previous considerations (see equation 3-49) we know that the equilibrium state given by a constant entropy, i.e. $S^{\mu}{}_{|\mu} = 0$, implies the existence of a time-like Killing vector field. But it is known that a Robertson-Walker type Universe does not admit a time-like Killing vector field. Thus it seems as if we have some sort of a paradox. But in actuality there is no paradox because of the fact that even though we don't have any shearing or heat conduction we still have bulk viscosity, and thus we have a dissipative agent. To see

this more explicitly, let us consider our universe in question to be described by the Robertson-Walker metric,

$$(5\text{-}1) \qquad ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2\right] .$$

Since the above line element is written in the comoving coordinate system, we thus have $u^1 = u^2 = u^3 = 0$, and $u^4 = 1$ for velocity $\underset{\sim}{u}$. Since we have no shearing or heat conduction, we see that in the comoving frame the Energy-Momentum tensor given in equation 4-13 becomes

$$T^{\mu\nu} = T_o^{\ \mu\nu} - \kappa\, g^{\mu\nu}\, u^\lambda{}_{|\lambda} = \mu\, u^\mu u^\nu + (P - \kappa u^\lambda{}_{|\lambda})\, \Delta^{\mu\nu} .$$

From which we deduce that

$$(5\text{-}2) \qquad T^{44} = \mu , \quad \text{and}$$

$$(5\text{-}3) \qquad T^{ij} = (P - \kappa\, u^\lambda{}_{|\lambda}) g^{ij} = (P - \kappa \cdot \Gamma^\alpha_{4\alpha}) g^{ij} \quad ,$$

where $i,j = 1,2,3$.

In view of equation 5-1, equation 5-3 yields

$$(5\text{-}4) \qquad T^{ij} = (P - 3\kappa \frac{\dot{R}}{R})\, g^{ij} \quad .$$

But we know that in the early "hot" stage of the universe, and the later "cold" stage of the universe κ vanishes. So that in

these stages $T^{\mu\nu}$, as expressed by equation 5-4, is just the Energy-Momentum tensor of a perfect fluid. But in the intermediary stages κ is non-vanishing and hence $T^{\mu\nu}$ is an imperfect fluid, which in turn means entropy production.

As a result of the above we are led to the question of Helium production. We know that Helium production takes place at temperatures of the order of 10^9K . This temperature is intermediate between the ultrarelativistic and the classical ranges. But if one examines equation 5-4 it can be seen that the bulk viscosity opposes the pressure. The question then to ask is: "Can the bulk viscosity counter act the pressure enough to slow down the expansion rate so that it spends a slightly longer time around $T \sim 10^9$K and thus produces a significant amount of more helium than would have been produced otherwise?". There are essentially two main reasons why one would expect the answer to be no. Firstly the pressures are so large, and the expansion rate $(\dot{R}/R)$ so small that bulk viscosity couldn't affect the expansion to any appreciable extent. The second reason is that the measured abundance of Helium is in close agreement with theoretical predictions which neglect bulk viscosity (3). One should note that this second reason given is not so devastating. The reason being that these measured abundances are gotten from "new" star populations. Accurate measurements have not yet been made for the old stars in the galactic halo. The abundance in these old stars is closer to the amount of helium produced cosmologically.

Another interesting aspect of bulk viscosity is whether it can account for the present entropy per baryon. Weinberg (4) has shown by

rather general arguments that bulk viscosity could not have given rise to the present entropy per Baryon for the present microwave background, in a Robertson-Walker type universe. What he essentially does is to first compute the rate of increase of the entropy per particle through the thermodynamical relation

$$T\,ds = du + P\,d\,v \,, \tag{5-5}$$

where v is the comoving volume, s , and u are the total entropy, and the total energy respectively in the comoving frame. From equation 5-5 we get

$$T\,\dot{s} = \dot{u} + P\,\dot{v} \quad , \tag{5-6}$$

where the dot denotes the derivative with respect to time. By means of Einsteins field equations one can prove that

$$\frac{d}{dt}(\mu R^3) = -3\,R^2\dot{R}\,(P - 3\kappa\,\frac{\dot{R}}{R}) \quad , \tag{5-7}$$

and

$$\frac{d}{dt}(nR^3) = 0 \quad , \tag{5-8}$$

where n is the density of particles.

If σ is the entropy per particle then the total energy s is

(5-9) $$s = n\, k\, \sigma\, R^3 \; .$$

The total energy u is

(5-10) $$u = \mu\, R^3$$

and the volume v is

(5-11) $$v = R^3 \; .$$

In view of equations 5-7, 8, 9, 10, 11 equation 5-6 becomes

(5-12) $$\dot{\sigma} = \frac{9\, \kappa\, \dot{R}^2}{n\, k\, T\, R^2} \; .$$

If the bulk viscosity arises from some form of radiation then to first order in τ , the mean free time, equation 5-12 reduces to (5),

(5-13) $$\dot{\sigma} = \frac{P\tau}{RT} \left[\frac{d}{dt}(RT)\right]_{\text{adiabatic}}$$

where P is the <u>adiabatic</u> rate of change of the radiation entropy; i.e.

(5-14) $$P \equiv \left[\frac{d}{dt}\left(\frac{4bT^3}{3\, n\kappa}\right)\right]_{\text{adiabatic}} \; .$$

Then by some general arguments he establishes limits to $\dot{\sigma}$. The result he gets is that (6)

(5-15) $$\dot{\sigma} \leq |P| \; ,$$

i.e. "the rate of entropy will always be less than if the radiation or matter gained entropy at the rate $|\dot{P}|$ and the matter or radiation had fixed entropy". (7). Weinberg thus arrives at two conditions that must be satisfied if there is to be an appreaciable production of entropy in a homogeneous isotropic expansion:

(a) "The matter and radiation must both contribute an appreciable share of the total energy. If either the radiation entropy or the matter entropy contributes only a small fraction of the total entropy throughout some time interval, then in an adiabatic expansion the change of radiation entropy would have to be a small fraction of the total entropy, so that the growth of entropy could at most be a small fraction of the total entropy" (8).

(b) We see that in view of equation 5-15 equation 5-13 implies that for maximum entropy growth, we must have (9),

$$\tau^{-1} \approx \left| \frac{1}{RT} \frac{d}{dt} (RT) \right| .$$

Weinberg then shows that for various kinds of radiations i.e. photons, electron-type neutrinos, muon-neutrinos, and gravitons, the above conditions aren't met. Which in turn leads Weinberg to the result that bulk viscosity could not account for the large entropy per baryon of the present microwave background.

For a detailed verification of Weinbergs conclusions, one would have to approach the problem from a kinetic theory point of view. Clearly the kinetic theory formalism as it is discussed in this thesis would be to naive to describe the above problem. One would have to formulate a theory for mixtures, i.e. allowing different species of particles. In formulating this theory for mixtures one would to allow for more general forms of interactions than close binary collisions. One would have to allow collisions in which particle number is not conserved. One would have to also allow multiple collisions (more than binary). There are also two processes that must be accounted for if the theory is to be a physically meaningful one: the first is Bremsstrahlung, and the second is the creation and annihilation of particles.

Finally we come to the question of the validity of the Relativistic Boltzmann equation. In the case of the non-relativistic Boltzmann equation, it can be shown that it is derivable from the more basic Liouville equation after a series of approximations. Some authors (10) have questioned the validity of the Relativistic Boltzmann equation because it has yet to be derived from the more basic equations. But this in turn raises the very fascinating question as to whether one can formulate a Relativistic version of Liouvilles theorem for interacting particles, and thus give rise to a Relativistic Statistical Mechanics.

FOOTNOTES

Chapter I

(1) Of course we must employ the usual assumption about $d^3r d^3v$, i.e. that it is large enough to contain a large number of particles, and yet small enough to be insignificant with respect to macroscopic dimensions. We also assume that the number of particles is large enough, and the variation of N is not great between successive elemental volumes, so that we can replace everything by continuous functions, and summations by integrations.

(2) Actually we are looking for quite a chaotic or unstructured system. In fact the most chaotic!

(3) We are of course assuming that the cells all have an equal apriori probability of being filled by any one particle.

(4) See Tolman "Statistical Mechanics".

(5) See J.L. Synge, "Relativity: The Special Theory", North Holland Publishing Company, 1965, p. 67.

(6) See Appendix A.

(7) $$s|t_r n^r| = \bar{s}|t_r \bar{n}_r| \ .$$

"This formula connects the 3-volumes $s, \bar{s}$ of cross sections normal to n^r, $\bar{n}^r$ respectively, of a world tube, t_r being any tangent vector to the tube."

See J.L. Synge "Relativity: The Special Theory", Chapter 8, section 6, North Holland Publishing Company, 1965.

(8) W. Israel, J. of Math. Phys. 1963 #9, p. 1164.

(9) J.L. Synge "The Relativistic Gas", Chapter 4, North Holland Publishing Company.

Chapter II

(1) Liouvilles Theorem.

(2) See p. 2-2.

(3) For proof see W. Israel, J. of Math. Phys., Vol. 4, #9, p. 1180.

(4) For proof see W. Israel, J. of Math. Phys., Vol. 4, #9, p. 1180.

(5) It should be noted that the expression for the scattering cross section, $\sigma(g,\Theta)$, that appears in equation 2-41 differs from the conventional scattering cross section given in physics. The reason for this lies in what we choose to call the relative 3-velocity in

the center of mass frame. The usual 3-velocity is $v_i = \frac{dx_i}{dt}$, but it turns out that in relativity theory a more natural choice for the relative 3-velocity is gotten by considering the 4-vector $\lambda_\mu = \frac{dx_\mu}{ds}$, and then considering the spatial components, λ_i, (i=1,2,3), of λ_μ. It can be shown (see Synge - "Relativity: The Special Theory" p. 130) that

$$\frac{v_i}{\alpha} = \frac{c\,\frac{\lambda_i}{2}}{(1 + \frac{\lambda^2}{4})^{1/2}} \quad .$$

Thus letting $g_\mu = \lambda_\mu$, $v = |\tilde{v}|$ we see that

$$\frac{v}{c} = \frac{g}{(1 + \frac{g^2}{4})^{1/2}} \quad .$$

To relate $\sigma(g,\Theta)$ to the conventional cross section $\sigma(v,\Theta)$, we choose the time axis to be along n^μ, and thus in the center of mass frame, n^μ and $\overline{P}^\mu$ are parallel. Then it follows from 1-19 that

$$d\Omega = d\,P_1\, dP_2\, dP_3 = d^3P = m \cosh\,\chi\; d\omega$$

and

$$d'\Omega = d^{3\prime}P = m \cosh \chi\; d'\omega \quad .$$

But $$\cosh \chi = (1 - \frac{v^2}{4c^2})^{-1/2} = (1 + \frac{1}{4}\, g^2)^{1/2} \quad .$$

Therefore expression 2-41 becomes

$$g\, \sigma(g,\Theta)\; N'N\; (1 - \frac{v^2}{4c^2})^{1/2}\; d^3P\; (1 - \frac{v^2}{4c^2})^{1/2}\; d^{3\prime}P\; d\tau$$

$$= v\, \sigma(g,\Theta)\; (1 + \frac{1}{4}\, g^2)^{-1/2}\; {}'N\, N\;\; d^3P\; d^{3\prime}P\;\; d\tau \quad .$$

Thus we see that

$$\sigma(g,\Theta)\ (1 + \frac{1}{4}\ g^2)^{-1/2} = \sigma(v,\Theta)$$

so that we get the conventional expression for 2-41, i.e.

$$v\ \sigma(v,\Theta)\ N'N\ d^3P P^3\ P'\ d\tau \ .$$

Chapter III

(1) For the more subtle statistical notions see;

D. Ter Haar, "Foundations of Statistical Mechanics", Reviews of Modern Physics, Vol. 27, No. 3, Section A.

Chapter IV

(1) For example u^μ could be defined in terms of the particle stream, i.e. u^μ parallel to M^μ, or in terms of the energy flow.

(2) There are many time scales involved in a gas. There is a Kinetic time scale, τ_K, a Hydrodynamic time scale, τ_H. The Kinetic time scale tells us how long the gas takes to get to a local equilibrium. The Hydrodynamic time scale gives us a characteristic time for a global smoothing out by Hydrodynamic effects. We will be concerned with times in the interval $\tau_K < \tau < \tau_H$, i.e. before τ_K Hydrodynamics is not applicable.

(3) Eckart - Physical Review, 58, p. 919 (1940).

(4) Weinberg - Astrophysical Journal Vol. 168, No. 2 page 1, Sept., 1971.

(5) For the non relativistic discussion of this see, Landau and Lifshitz, "Fluid Mechanics" (Addison-Wesley 1959).

For the relativistic discussion of this see, Weinberg Astrophysical Journal, Vol. 168, No. 2, Page 1, Sept. 1971.

(6) See Weinberg, Ap. J. Vol. 168, No. 2, page 1, Sept. 1971.

(7) See Israel, Journal of Mathematical Physics, Vol. 4, No. 9, p. 1163, Sept. 1963.

To see that the two definitions are equivalent see Appendix B.

(8) See Israel, Journal of Mathematical Physics, Vol. 4, No. 9, Sept. 1963, equation 9-12.

(9) Ibid, equation 9-13.

For the explicit calculation see Appendix C.

(10) For classical discussion of Grad Method see Grad. Hb. d. Phys. Volume 12, S. Flügge ed. (Springer, Berlin 1958).

(11) Anderson, "Proceedings of the Relativity Conference in the Midwest", Edited by Carmeli, Fickler, and Witten (Plenun Press 1970), page 109.

(12) Ibid, see page 117.

(13) Ibid, see page 119.

(14) See Appendix D.

(15) Israel, J. of Math. Phys., Vol. 4, No. 9, Sept. 1963, equation 6.28.

(16) Ibid, equations 6.35, and 6.36.

(17) Ibid, equation 8.14.

(18) Ibid, equation 8.15.

(19) Ibid, equations 6.29, and 8.16.

(20) Consider arbitrary $b^{\alpha\beta}$, let $b \equiv g^{\alpha\beta} b_{\alpha\beta}$ then

$$b^{\alpha\beta} = (b^{\alpha\beta} - \frac{1}{4} g^{\alpha\beta} b) + \frac{1}{4} g^{\alpha\beta} b \equiv B^{\alpha\beta} + \frac{1}{4} g^{\alpha\beta} b \quad .$$

It follows then that

$$b^{\alpha\beta} P_\alpha P_\beta + C_5 = B^{\alpha\beta}(P_\alpha P_\beta) + \frac{1}{4} b(q^{\alpha\beta} P_\alpha P_\beta) + C_5 =$$

$$= B^{\alpha\beta} P_\alpha P_\beta + C'_5 \quad ,$$

where $C'_5 = \frac{1}{4} b(-m^2) + C_5$ and $g^{\alpha\beta} B_{\alpha\beta} = 0$.

Thus we can absorb the trace of $b_{\alpha\beta}$ into our constant, so that we can without loss of generality let $b_{\alpha\beta} g^{\alpha\beta} = 0$.

(21) See Appendix E.

(22) Israel, J. of Math. Phys., Vol. 4, No. 9, Sept. 1963, equation 6.37.

(23) Ibid, equations 6.35, and 6.36.

(24) Ibid, equation 6.37.

Chapter V

(1) For detailed calculations see Appendix F.

(2) For more discussion on this matter see;
Shucking, Spiegel, "Comments on Astrophysics and Space Physics", Vol. 2, p. 121 (1970).
Stewart, MacCallum, and Sciama, "Comments on Astrophysics and Space Physics", Vol. 2, p. 206.
Anderson, "Proceedings of the Relativity Conference in the Midwest" published by Plenum Press, 1970, page 109.

(3) Peebles, R.A.S.C. Journal, Vol. 63, No. 1, page 5.

(4) Weinberg, Ap. Journal, Vol. 168, No. 2, page 1, Sept. 1971.

(5) Ibid, see equation 3.14.

(6) Ibid, see equation 3.17.

(7) Ibid.

(8) Ibid.

(9) Ibid, see equation 3-19.

(10) Shucking, and Spiegel, "Comments on Astrophysics and Space Physics", Vol. 2, p. 121 (1970).

APPENDIX A

We've already seen that

A-1 $$\nu = N \, ds \, d\Omega \quad ,$$

where ds and $d\Omega$ are orthogonal to some timelike unit vector n_μ . If we choose n_μ to be along the time axis then

A-2 $$ds = dx_1 \, dx_2 \, dx_3$$

A-3 $$d\Omega = d\,P_1 \, d\,P_2 \, d\,P_3 \quad .$$

Equation A-3 is a volume element in the momentum space. If we wish to deal in velocity space then the volume element is

A-4 $$d\mathcal{U} = d\,u_1 \, d\,u_2 \, d\,u_3 \quad .$$

Then it can be shown [see J.L. Synge "The Relativistic Gas", p. 16, North Holland Pub.] that

A-5 $$d\Omega = \frac{m^3 \gamma^5}{c^3} \, d\mathcal{U} \quad , \quad \text{where} \quad \gamma = (1 - \frac{u^2}{c^2})^{-1/2} \quad .$$

Equation A-1 now becomes

A-6 $$\nu = \frac{N \, m^3}{c^3} \gamma^5 \, dx_1 \, dx_2 \, dx_3 \, du_1 \, du_2 \, du_3 \quad .$$

Equation A-6 tells the number of particles found in a box $dx_1\ dx_2\ dx_3$ of the observers space at time t, with velocities in the range $du_1\ du_2\ du_3$.

As can be seen that because of the γ^5 term the distribution is weighted towards the high speeds. This rather "unnatural" result disappears if we stay in Momentum space.

Another reason why velocity space is unfavourable is because if we ever want to treat photons it couldn't be done in velocity space, because they all have the same speed c and therefore there are no ranges to speak of.

APPENDIX B

Starting with

B-1 $$Q^{\mu} = \lambda\ \Delta^{\mu\nu}\theta_{,\nu}$$

and using a result given by Israel [J. of Math. Phys. Vol. 4, No. 9, Sept. 1963, equation 6.29]

B-2 $$\theta_{,\nu} = \frac{k\eta}{mc^2}\beta_{,\nu} + \frac{k\eta\beta}{mc^4}\frac{1}{(\mu + \frac{P}{c^2})}P_{,\nu}$$

But $\beta = \frac{mc^2}{k\ T}$ therefore $\beta_{,\nu} = -\frac{mc^2}{kT^2}T_{,\nu}$ so that equation B-2 becomes

B-3 $$\theta_{,\nu} = -\frac{\eta}{T^2}T_{,\nu} + \frac{\eta}{c^2T}\frac{1}{(\mu + \frac{P}{c^2})}P_{,\nu} .$$

But we've already seen that (see equation 3-65)

B-4 $$\dot{u}_{\mu} = -\frac{1}{(\mu + \frac{P}{c^2})}\Delta^{\nu}_{\ \mu}P_{,\nu} .$$

Substituting equation B-4 into equation B-3 and then substituting equation B-3 into equation B-1 we get

$$Q^{\mu} = \lambda\Delta^{\mu\nu}\left(-\frac{\eta}{T^2}T_{,\nu} - \frac{\eta}{c^2T}\dot{u}_{\nu}\right)$$

which simplifies to

$$Q^{\mu} = - \left(\frac{\lambda\eta}{T^2}\right) \Delta^{\mu\nu} \left(T_{,\nu} + \frac{T}{c^2} \dot{u}_{\nu}\right)$$

or

B-5 $$Q^{\mu} = - \chi \, \Delta^{\mu\nu} \left(T_{,\nu} + T \, \dot{u}_{\nu}\right) \quad .$$

Equation B-5 is just the phenomenological form of the heat flux vector where

$$\chi = \frac{\lambda\eta}{T^2} \quad .$$

APPENDIX C

We know that

C-1 $$\frac{\partial}{\partial x^\nu} \int N_o \ \Psi(P)\ P^\nu \ d\omega = -\frac{1}{4} \iiiint W \ N_o \ 'N_o \ \delta(f)\delta(\Psi) \ d\omega \ d'\omega \ d\omega^* \ d'\omega^* .$$

Let

$$\Psi = P^\lambda P^\mu$$

then

C-2 $$\frac{\partial}{\partial x^\nu} \int N_o \ P^\lambda P^\mu P^\nu d\omega = -\frac{1}{4} \iiiint W \ N_o \ 'N_o \ \delta(f)\delta(P^\lambda P^\mu) \ d\omega \ d'\omega \ d\omega^* \ d'\omega^* .$$

Substituting $f = f_{Hom} + b_{\alpha\beta} \ P^\alpha P^\beta$ into equation C-2 we get

C-3 $$\mathcal{U}_o{}^{\lambda\mu\nu}{}_{,\nu} = -\frac{1}{4} mc^3 \iiiint W \ N_o \ 'N_o \ \delta(f_{Hom}) \ \delta(P^\lambda P^\mu) \ d\omega \ d'\omega \ d\omega^* \ d\,'\omega^* -$$

$$-\frac{1}{4} b_{\alpha\beta} \iiiint W \ N_o \ 'N_o \ \delta(P^\alpha P^\beta)\delta(P^\lambda P^\mu) \ d\omega \ d'\omega \ d\omega^* \ d'\omega^* .$$

But $\delta(f_{Hom}) = 0$, therefore

$$\mathcal{U}_o{}^{\lambda\mu\nu}{}_{|\nu} = - b_{\alpha\beta} \ X^{\alpha\beta\lambda\mu} \qquad \text{where}$$

$$X^{\alpha\beta\lambda\mu} = \frac{1}{4} mc^3 \iiiint W \ N_o \ 'N_o \ \delta(P^\alpha P^\beta)\delta(P^\lambda P^\mu) \ d\omega \ d'\omega \ d\omega^* \ d'\omega^* .$$

APPENDIX D

For the sake of calculation let $c = 1$. To first order in f ,

D-1
$$-M_\alpha M^\alpha = -(M_{o\alpha} + M_{1\alpha})(M_o^{\ \alpha} + M_1^{\ \alpha})$$
$$= -M_{o\alpha}M_o^{\ \alpha} - M_1^{\ \alpha}M_{o\alpha} - M_{1\alpha}M_o^{\ \alpha} = \rho^2$$

$\therefore$

$$Q^\mu = -\frac{1}{\rho}(T_o^{\ \lambda\nu} + T_1^{\ \lambda\nu})(M_{o\nu} + M_{1\nu})[\delta^\mu_{\ \lambda} + \frac{1}{\rho^2}(M_{o\lambda} + M_{1\lambda})(M_{o\lambda} + M_1^{\ \mu})]$$
$$= -\frac{1}{\rho}(T_o^{\ \lambda\nu} + T_1^{\ \lambda\nu})(M_{o\nu} + M_{1\nu})[\delta^\mu_{\ \lambda} + \frac{1}{\rho^2}(M_{o\lambda}M_o^{\ \mu} + M_{1\lambda}M_o^{\ \mu} + M_{o\lambda}M_o^{\ \mu})]$$
$$= -\frac{1}{\rho}\{(T_o^{\ \lambda\nu}M_{o\nu} + T_o^{\ \lambda\nu}M_{1\nu} + T_1^{\ \lambda\nu}M_{o\nu})[\delta^\mu_{\ \lambda} + \frac{1}{\rho^2}(M_{o\lambda}M_o^{\ \mu} + M_{1\lambda}M_o^{\ \mu} + M_{o\lambda}M_1^{\ \mu})]\}$$
$$= -\frac{1}{\rho}\{(T_o^{\ \lambda\nu}M_{o\nu} + T_o^{\ \lambda\nu}M_{1\nu} + T_1^{\ \mu\nu}M_{o\nu}) + \frac{1}{\rho^2}(T_o^{\ \lambda\nu}M_{o\lambda}M_{o\nu}M_o^{\ \mu} +$$
$$+ T_o^{\ \lambda\nu}M_{1\nu}M_{o\lambda}M_o^{\ \mu} + T_1^{\ \lambda\nu}M_{o\nu}M_{o\lambda}M_o^{\ \mu} + T_o^{\ \lambda\nu}M_{o\nu}M_o^{\ \mu}M_{1\lambda} +$$
$$+ T_o^{\ \lambda\nu}M_{o\nu}M_{o\lambda}M_1^{\ \mu})\} .$$

But $T_o^{\ \mu\nu} = \mu\, u^\mu u^\nu + P\, \Delta^{\mu\nu}$ and $M_o^{\ \mu} = \rho u^\mu$, so that $T_o^{\ \mu\nu}M_{o\nu} = -\rho\,\mu\,\mathbf{u}^\mu$, $T_o^{\ \mu\nu}M_{1\nu} = P\,\Delta^{\mu\nu}M_{1\mu} = PM_1^{\ \mu}$, $T_1^{\ \mu\nu}M_{o\nu} = \rho\, T_1^{\ \mu\nu}\mathbf{u}_\nu$, $M_o^{\ \mu}T_o^{\ \lambda\nu}M_{o\nu}M_{o\lambda} = \rho^3\,\mu\,\mathbf{u}^\mu$, $T_o^{\ \lambda\nu}M_{1\nu}M_{o\lambda}M_o^{\ \mu} = P\,M_1^{\ \lambda}M_{o\lambda}M_o^{\ \mu} = 0$, and $T_o^{\ \lambda\nu}M_{o\nu}M_{o\lambda}M_1^{\ \mu} = \rho^2\,\mu\,M_1^{\ \mu}$

∴

$$Q^{\mu} = -\frac{1}{\rho}\{(-\rho\mu u^{\mu} + PM_1^{\ \mu} + \rho T_1^{\ \mu\nu}u_{\nu}) + \frac{1}{\rho^2}(\rho^3\mu u^{\mu} + \rho^2\mu\, M_1^{\ \mu})\}$$

$$Q^{\mu} = -\left(\frac{\mu+P}{\rho}\right) M_1^{\ \mu} - T_1^{\ \mu\nu}\, u_{\nu} \quad .$$

Noting that $\frac{\mu+P}{\rho} = \eta$ since $u_{\nu}u_{\rho}\, T_1^{\ \rho\nu} = 0$ then

$$\Delta^{\mu}_{\ \rho}\, T_1^{\ \rho\nu}\, u_{\nu} = T_1^{\ \mu\nu}\, u_{\nu}$$

∴

$$Q^{\mu} = -\,\eta\, M_1^{\ \mu} - \Delta^{\mu}_{\ \rho}\, T_1^{\ \rho\nu}\, u_{\nu} \quad .$$

APPENDIX E

E-1 $$v_o^{\alpha\beta\mu\nu} = mc^4 \frac{\partial^4 z_o}{\partial\beta_\alpha \partial\beta_\beta \partial\beta_\mu \partial\beta_\nu}$$

$$z_o(\alpha, \beta_\mu) = 4\pi\ \alpha\ \frac{K_1(\beta)}{\beta}$$

$$\frac{\partial z_o}{\partial \beta_\nu} = \frac{\partial z_o}{\partial \beta} \frac{\partial\ \beta}{\partial\beta_\nu} = 4\pi\ \alpha\ \frac{K_2(\beta)}{\beta^2}\ \beta^\nu$$

where

$$\frac{d}{d\beta}\left[\beta^{-n} K_n(\beta)\right] = -\ \beta^{-n} K_{n+1}\ (\beta)$$

$$\frac{\partial\ \beta}{\partial\beta_\mu} = -\ \frac{\beta^\mu}{\beta}\ , \quad \text{and} \quad \frac{\partial\beta^\mu}{\partial\beta_\nu} = g^{\mu\nu}$$

$$\frac{\partial^2 z_o}{\partial\beta_\mu \partial\beta_\nu} = 4\pi\ \alpha\ \frac{K_3(\beta)}{\beta^3}\ \beta^\nu\beta^\mu + 4\pi\ \alpha\ \frac{K_2(\beta)}{\beta^2}\ g^{\mu\nu}\ .$$

Similarly

E-2 $$\frac{\partial^4 z_o}{\partial\beta_\alpha \partial\beta_\beta \partial\beta_\mu \partial\beta_\nu} = 4\pi\ \alpha\ \frac{K_5(\beta)}{\beta^5}\ \beta^\alpha\beta^\beta\beta^\nu\beta^\mu + 4\pi\ \alpha\ \frac{K_4(\beta)}{\beta^4}$$

$$[\beta^\nu\beta^\mu g^{\alpha\beta} + \beta^\nu\beta^\beta g^{\mu\alpha} + \beta^\mu\beta^\beta g^{\nu\alpha} + \beta^\alpha\beta^\nu g^{\mu\beta} + \beta^\mu\beta^\alpha\ {}^{\nu\beta}$$

$$+ \beta^\alpha\beta^\beta g^{\mu\nu}] + 4\pi\ \alpha\ \frac{K_3(\beta)}{\beta^3}\ [g^{\nu\alpha}g^{\mu\beta} + g^{\nu\beta}g^{\mu\alpha} + g^{\alpha\beta}g^{\mu\nu}]\ .$$

Noting the identities

E-3
$$g^{\nu\alpha}g^{\mu\beta} + g^{\nu\beta}g^{\mu\alpha} + g^{\alpha\beta}g^{\mu\nu} \equiv \Delta^{\nu\alpha}\Delta^{\mu\beta} + \Delta^{\nu\beta}\Delta^{\mu\alpha} + \Delta^{\alpha\beta}\Delta^{\mu\nu}$$
$$- g^{\nu\alpha}\frac{u^{\mu}u^{\beta}}{c^2} - g^{\mu\beta}\frac{u^{\nu}u^{\alpha}}{c^2} - g^{\nu\beta}\frac{u^{\mu}u^{\alpha}}{c^2} - g^{\mu\alpha}\frac{u^{\nu}u^{\beta}}{c^2} -$$
$$- \frac{u^{\nu}u^{\mu}}{c^2}g^{\alpha\beta} - \frac{u^{\alpha}u^{\beta}}{c^2}g^{\mu\nu} - 3\,\frac{u^{\alpha}u^{\beta}u^{\mu}u^{\nu}}{c^4}$$

E-4
$$u^{\nu}u^{\mu}\Delta^{\alpha\beta} + u^{\alpha}u^{\beta}\,{}^{\mu\nu} + 4\,u^{(\nu}\Delta^{\mu)(\alpha}u^{\beta)} - 6\,u^{\alpha}u^{\beta}u^{\mu}u^{\nu} =$$
$$= u^{\nu}u^{\mu}g^{\alpha\beta} + u^{\nu}u^{\beta}g^{\mu\alpha} + u^{\mu}u^{\beta}g^{\nu\alpha} + u^{\alpha}u^{\nu}g^{\mu\beta} + u^{\mu}u^{\alpha}g^{\nu\beta} + u^{\alpha}u^{\beta}g^{\mu\nu} \;.$$

Equation E-2 simplifies to

E-5
$$v_o^{\alpha\beta\mu\nu} = 4\pi\, m\alpha\left\{\frac{K_5(\beta)}{\beta} - \frac{6K_4(\beta)}{\beta^2} + \frac{3K_3(\beta)}{\beta^3}\right\} u^{\alpha}u^{\beta}u^{\mu}u^{\nu}$$
$$+ 4\pi\, m\alpha c^2 \left\{\frac{K_4(\beta)}{\beta^2} - \frac{K_3}{\beta^3}\right\} 4\,u^{(\nu}\Delta^{\mu)(\alpha}u^{\beta)} +$$
$$+ 4\pi\, m\alpha c^4\left\{\frac{K_3(\beta)}{\beta^3}\right. \{\Delta^{\nu\alpha}\Delta^{\mu\beta} + \Delta^{\nu\beta}\Delta^{\mu\alpha} + \Delta^{\alpha\beta}\Delta^{\mu\nu}\} +$$
$$+ 4\pi\, m\alpha c^2\left\{\frac{K_4}{\beta^2} - \frac{K_3}{\beta^3}\right\} \{u^{\alpha}u^{\beta}\Delta^{\mu\nu} + u^{\mu}u^{\nu}\Delta^{\alpha\beta}\} \;.$$

APPENDIX F

We know that

F-1 $$\kappa = \left(\frac{P\eta\Omega}{c}\right)^2 \frac{1}{A\beta} \qquad \Omega = 3\gamma - 5 + \frac{3\gamma}{\eta\beta} \quad , \quad \eta = \frac{K_3}{K_2} \quad .$$

Let us consider $\beta \approx 0$, i.e. the ultrarelative limit $(T \to \infty)$. For $\beta \approx 0$ we know that to 2nd order

F-2 $$K_3(\beta) \approx \frac{8}{\beta^3} - \frac{1}{\beta} = \frac{8-\beta^2}{\beta^3}$$
$$K_2 \approx \frac{2}{\beta^2} - \frac{1}{2} = \frac{4-\beta^2}{2\beta^2}$$

so that we have

$$\eta = \frac{K_3}{K_2} \approx \frac{8-\beta^2}{\beta^3} \; \frac{2\beta^2}{4-\beta^2} \;\; = \;\; \frac{2}{\beta}\,(8-\beta^2)\,\frac{1}{4}\left(1 - \frac{\beta^2}{4}\right)^{-1}$$
$$\approx \frac{1}{2\beta}\,(8-\beta^2)\left(1 + \frac{\beta^2}{4}\right) \approx \frac{4}{\beta} + \frac{\beta}{2} \quad .$$

But

$$\frac{\gamma}{\gamma-1} = -\,\beta^2\,\frac{d\eta}{d\beta} = -\,\frac{4}{\beta^2} + \frac{1}{2} \Rightarrow \gamma \approx \frac{4}{3} + \frac{\beta^2}{18} \qquad \text{for} \qquad \beta \approx 0 \quad .$$

So that for $\beta \approx 0$, $\Omega = 3\gamma - 5 + \frac{3\gamma}{\eta\beta}$, but $\eta\beta \approx 4 + \frac{\beta^2}{2}$, thus we have

$$\Omega \approx 3\left(\frac{4}{3} - \frac{\beta^2}{18}\right) - 5 + \frac{3\left(\frac{4}{3} + \frac{\beta^2}{18}\right)}{4 + \beta^2/2}$$

$$\Omega \approx 4 - \frac{\beta^2}{6} - 5 + \left(4 + \frac{\beta^2}{8}\right)\frac{1}{4}\left(1 + \frac{\beta^2}{8}\right)^{-1}$$

F-4 $$\Omega \approx -1 - \frac{\beta^2}{6} + \frac{1}{4}\left(4 + \frac{\beta^2}{8}\right)\left(1 - \frac{\beta^2}{8}\right)$$

$$\Omega \approx -1 + \frac{\beta^2}{6} + 1 - \frac{\beta^2}{12}$$

$$\Omega \approx \frac{\beta^2}{12}$$

It should be noted that we were forced to take K_3 , K_2 to 2nd order (i.e. up to the 2nd term in series expansion) in calculating Ω because of the factor β which was multiplying η in this expression for η .

We also know that

$$P\eta = 4\pi\ m\alpha c^2 \frac{K_2}{\beta^2} \frac{K_3}{K_2} = 4\pi\ m\alpha c^2 \frac{K_3}{\beta^2} ,$$

for $\beta \approx 0$

F-5 $$P\eta \approx 32\pi\ m\alpha c^2 \left(\frac{1}{\beta^5}\right) .$$

Let us next calculate A for $\beta \approx 0$:

F-6 $$A = \frac{8\pi^3 m\alpha^2 c^{-1}}{\beta} \iint \sin^3\Theta \frac{K_3(y)}{y^2} g^7 \sigma(g,\Theta)\ dg\ d\Theta ,$$

where

$$y = 2\beta \left(1 + \frac{1}{4} g^2\right)^{1/2} ,$$

We know that

$$\sigma(g,\Theta) = \sigma(v,\Theta) \left(1 + \frac{1}{4} g^2\right)^{1/2} \quad ; \quad \text{or}$$

$$\sigma(g,\Theta) = \frac{y\sigma(v,\Theta)}{2\beta} ,$$

so that equation F-5 becomes

F-7 $$A = \frac{8\pi^3 m\alpha^2 c^{-1}\tilde{\sigma}}{2\beta^2} \iint \sin^3\Theta\ \sigma(v,\Theta)\ d\Theta\ g^7 \frac{K_3(y)}{y} dg .$$

Near $\beta \approx 0$ that is $T \to \infty$ we will assume that

$$\sigma(v,\Theta) = \sigma(\Theta)$$

so that equation F-6 becomes

F-8
$$A = \frac{8\pi^3 m\alpha^2 c^{-1}}{2\beta^2} \int \frac{K_3(y)}{y} g^7 \, dg$$

where

$$\tilde{\sigma} = \int \sin^3\Theta \, \sigma(\Theta) \, d\Theta \ .$$

Consider the integral,

F-9
$$W = \int_0^\infty \frac{K_3(y)}{y} g^7 \, dg$$

using $y = 2\beta \left(1 + \frac{1}{4} g^2\right)^{1/2}$ we transform the variable of integration to y i.e.

$$y^2 = 4\beta^2\left(1 + \frac{1}{4} g^2\right) \longrightarrow g^2 = \frac{y^2 - 4\beta^2}{\beta^2}$$

$$\longrightarrow g^6 = \frac{(y^6 - 12y^4\beta^2 + 48y^2\beta^4 - 64\beta^6)}{\beta^6}$$

and also that $\frac{y\,dy}{\beta^2} = g \, dg$, thus equation F-8 becomes

F-10
$$W = \frac{1}{\beta^8} \int_{2\beta}^\infty y^6 K_3(y) \, dy - \frac{12}{\beta^6} \int_{2\beta}^\infty y^4 K_3(y) \, dy$$

$$+ \frac{48}{\beta^4} \int_{2\beta}^\infty y^2 K_3(y) \, dy - \frac{64}{\beta^2} \int_{2\beta}^\infty K_3(y) \, dy \ .$$

Consider W term by term we have:

$$I = \frac{1}{\beta^8}\int_{2\beta}^{\infty} y^6\, K_3(y) = \frac{1}{\beta^8}\int_0^{\infty} K_3(y)\, y^6\, dy - \frac{1}{\beta^8}\int_0^{2\beta} y^6\, K_3\, dy \quad ,$$

the integral $c_1 = \int_0^\infty y^6\, K_3\, dy$ is bounded so that we can write

$$I = \frac{c_1}{\beta^8} - \frac{1}{\beta^8}\int_0^{2\beta} K_3\, y^6\, dy \quad .$$

But in the range $(0,2\beta)$

$$K_3(y) \approx \frac{8}{y^3} \quad ,$$

thus

F-11 $$I = \frac{c_1}{\beta^8} - \frac{8}{\beta^8}\int_0^{2\beta} y^3\, dy = \frac{c_1}{\beta^8} + 0(\frac{1}{\beta^4}) \quad .$$ For

$$II = -\frac{12}{\beta^6}\int_{2\beta}^{\infty} K_3 y^4\, dy = -\frac{12}{\beta^6}\int_0^{\infty} y^4\, K_3\,(y)\, dy$$

$$+ \frac{12}{\beta^6}\int_0^{2\beta} K_3\, y^4\, dy \quad .$$

By similar arguments we get

F-12 $$II = \frac{c_2}{\beta^6} + 0(\frac{1}{\beta^4}) \quad ,$$

where $c_2 = -12\int_0^{\infty} y^4\, K_3\, dy$. For

$$III = \frac{48}{\beta^4}\int_{2\beta}^{\infty} y^2\, K_3\, dy \quad ,$$

we can write

$$III = \frac{48}{\beta^4} \int_{2\beta}^{\infty} y^2 K_3 \, dy = \frac{48}{\beta^4} \int_{2\beta}^{\infty} (y^2 K_3 - \frac{8}{y}) dy + \frac{48}{\beta^4} \int_{2\beta}^{\infty} \frac{8}{y} \, dy$$

which can also be rewritten as

$$III = \frac{48}{\beta^4} \int_0^{\infty} (y^2 K_3 - \frac{8}{y}) dy - \frac{48}{\beta^4} \int_0^{2\beta} (y^2 K_3 - \frac{8}{y}) \, dy + \int_{2\beta}^{\infty} \frac{8}{y} \, dy \; .$$

The integral $c_3 = \int_0^{\infty} (y^2 K_3 - \frac{8}{y}) \, dy$ is bounded, and in the interval $(0,2\beta)$ we once again can approximate $K_3(y) \approx \frac{8}{y^3}$, therefore integral III becomes

F-13 $$III = \frac{c_3}{\beta^4} + 0(\frac{1}{\beta^2}) \; .$$

The integral $IV = - \frac{64}{\beta^2} \int_{2\beta}^{\infty} K_3 \, dy$ can be rewritten as

$$IV = - \frac{64}{\beta^2} \int_{2\beta}^{\infty} (K_3 - \frac{8}{y^2} + \frac{1}{y}) \, dy - \frac{64}{\beta^2} \int_{2\beta}^{\infty} (\frac{8}{y^2} - \frac{1}{y}) \, dy \; ,$$

but this in turn can be rewritten as

$$IV = - \frac{64}{\beta^2} \int_0^{\infty} (K_3 - \frac{8}{y^2} + \frac{1}{y}) + \frac{64}{\beta^2} \int_0^{2\beta} (K_3 - \frac{8}{y^2} + \frac{1}{y}) \, dy \; -$$

$$- \frac{64}{\beta^2} \int_{2\beta}^{\infty} (\frac{8}{y^2} - \frac{1}{y}) \, dy \; .$$

By arguments similar to those given for the integral II we get,

F-14 $$IV = - \frac{c_4}{\beta^2} + 0(\frac{1}{\beta^5}) \; .$$

Using the results given in equations F-10, 11, 12, 13, equation F-9 becomes

$$W = \frac{c_1}{\beta^8} + 0(\frac{1}{\beta^4}) + \frac{c_2}{\beta^6} + 0(\frac{1}{\beta^4}) + 0(\frac{1}{\beta^4}) + 0(\frac{1}{\beta^5}) \quad .$$

Thus we finally get

$$W = \frac{c_1}{\beta^8} + 0(\frac{1}{\beta^6}) \qquad \text{or,}$$

F-15
$$W \approx \frac{c_1}{\beta^8}$$

Upon substituting equation F-14 into equation F-7, and together with equations F-4, 5, equation F-1 becomes

F-16
$$\kappa \approx (\frac{16c^5m}{9\pi\tilde{\sigma}c_1})\ \beta^3$$